1. The differential equation
\[ x \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} + ny = 0 \]
is known as Laguerre’s equation.

a) Obtain the regular solution in the form
\[
y_1(x) = B_0 \left[ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{n(n-1)(n-2) \cdots (n-k+1)}{(k!)^2} x^k \right]
\]

b) Show that this solution is a polynomial of degree \( n \) when \( n \) is a nonnegative integer, and verify that the choice \( B_0 = 1 \) leads to the Laguerre polynomial of degree \( n \), with the definition
\[
L_n(x) = 1 - \binom{n}{1} \frac{x}{1!} + \binom{n}{2} \frac{x^2}{2!} - \cdots + \frac{(-x)^n}{n!},
\]

where \( \binom{n}{k} \) represents the binomial coefficient \( n! / [(n-k)!k!] \).

2. The differential equation
\[ \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0 \]
is known as Hermite’s equation.

Obtain the general solution in the form \( y(x) = c_1 y_1(x) + c_2 y_2(x) \), where

\[
u_n(x) = x - \binom{n-1}{1} \frac{x^3}{1!} + \frac{(n-1)(n-3)}{3 \cdot 5} \frac{x^5}{2!} + \cdots.
\]

[Hence verify that the solution \( u_n(x) \) is a polynomial of degree \( n \) when \( n \) is a positive even integer or zero, whereas \( v_n(x) \) is a polynomial of degree \( n \) when \( n \) is a positive odd integer. That multiple of the \( n \)th-degree polynomial for which the coefficient of \( x^n \) is \( 2^n \) is called the \( n \)th Hermite polynomial and is often denoted by \( H_n(x) \).]