1. (a)  If $\alpha$ is a positive real constant, determine the Fourier sine and cosine integral representations of $e^{-\alpha x}$ in the forms

$$e^{-\alpha x} = \int_0^\infty \left( \frac{2}{\pi} \frac{u}{\alpha^2 + u^2} \right) \sin \alpha x \, du = \int_0^\infty \left( \frac{2}{\pi} \frac{\alpha}{\alpha^2 + u^2} \right) \cos \alpha x \, du \quad (\alpha > 0, x > 0).$$

(b)  Use these results to determine functions $A(u)$ and $B(u)$ such that

$$\frac{x}{\alpha^2 + x^2} = \int_0^\infty A(u) \sin \alpha x \, du, \quad \frac{\alpha}{\alpha^2 + x^2} = \int_0^\infty B(u) \cos \alpha x \, du \quad (\alpha > 0, x > 0).$$

(c)  Deduce from the preceding results that

(*Note: $S =$ Sine Fourier Transform, below, and $C=$Cosine Fourier Transform)

$$S\left\{ \frac{x}{\alpha^2 + x^2} \right\} = \frac{\pi}{2} e^{-\alpha x}, \quad S\{e^{-\alpha x}\} = \frac{u}{\alpha^2 + u^2} \quad \text{and}$$

$$C\left\{ \frac{\alpha}{\alpha^2 + x^2} \right\} = \frac{\pi}{2} e^{-\alpha x}, \quad C\{e^{-\alpha x}\} = \frac{\alpha}{\alpha^2 + u^2} \quad \text{when} \ \alpha > 0.$$  

2. Show that $\alpha$ can be replaced by $a + ib$, where $a$ and $b$ are real and $a > 0$, in Problem 1(a). In particular, show that

$$e^{-\alpha x} (\cos \beta x - i \sin \beta x) = \frac{2}{\pi} \int_0^\infty \frac{u \sin \alpha x}{(a^2 - b^2 + u^2) + 2iab} \, du$$

when $a > 0$ and $x > 0$, by equating real and imaginary parts of the equal members, deduce the sine and cosine integral representations of $e^{-\alpha x} \cos \beta x$ and $e^{-\alpha x} \sin \beta x$ when $a > 0$.

3. (a)  If $\alpha$ is a positive real constant, determine the complex form of the Fourier integral representation of the $e^{-\alpha |x|}$

$$e^{-\alpha |x|} = \int_{-\infty}^{\infty} \left( \frac{1}{\pi} \frac{\alpha}{\alpha^2 + u^2} \right) e^{iux} \, du \quad (\alpha > 0).$$

(b)  Deduce from this result the function $C(u)$ such that

$$\frac{\alpha}{\alpha^2 + x^2} = \int_{-\infty}^{\infty} C(u) e^{iux} \, du \quad (\alpha > 0).$$

(c)  Deduce that

(*Note: $F =$ Fourier IntegralTransform, below)

$$F\left\{ \frac{\alpha}{\alpha^2 + x^2} \right\} = \pi e^{-\alpha |x|}, \quad F\{e^{-\alpha |x|}\} = \frac{2\alpha}{\alpha^2 + u^2} \quad \text{when} \ \alpha > 0.$$