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The effect of longitudinal electric field components on the propagation of intense ultrashort optical pulses

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ABSTRACT

In this paper we investigate the effect of longitudinal electric field components on the propagation of intense ultrashort optical pulses. We find that the longitudinal electric field components have contributions both from modes that in the paraxial limit are polarized transversely to the beam axis and traveling along the beam axis and from a mode that is polarized along the beam axis even in the paraxial limit and that travel transversely to the beam axis. We show that the amplitude of this last mode in general satisfies a dispersive wave equation in the plane transverse to the beam axis and that the source term in this equation depends on the amplitude of both types of modes. The source is large if the fields are intense, the pulse is short or the fields are nonparaxial. Thus the effect of the mode that is polarized along the beam axis is to transport energy away from the collapse region and in this way make the system less prone to self-focusing collapse. In the weakly nonlinear slowly varying limit we show explicitly that the effect of the longitudinally polarized mode is to restrict the range of transverse modulationally unstable wavenumbers and to act as a defocusing lens in the collapse region.

1. Introduction

Ultrafast laser pulse experiments and applications are now entering a phase that challenge the validity of physical models utilized for longer pulses in nonlinear optics. When nonlinearity and strong self-focusing effects dominate, pulses can undergo strong compression down to a few cycles, potentially develop optical shock waves and generate strongly anisotropic non-equilibrium distributions of photo-ionized electrons and ions. Important applications of intense terawatt (TW) power femtosecond laser pulses include remote sensing, and efficient long distance energy transfer including confinement of radio frequency or microwave radiation. At petawatt (PW) power levels, the relativistically intense powers can accelerate electrons up to 1.45 GeV when focused down to many micron beam waists [1]. In the latter scenario, peak power densities approaching 10^{24} watts per centimeter squared are achievable. Ultrashort pulse propagation in general has been the focus of much theoretical and experimental work and several review articles of TW and sub-TW pulse propagation in gases and condensed media have been published by Couairon and Mysyrowicz [2], Chin et al. [3]. Berge et al. [4] and Kasparian et al. [5]. For a recent review of PW ultrashort laser pulse applications see the article by Korzhimanov et al. [6].

The qualitative picture emerging from earlier studies of lower power continuous wave and pulsed lasers is well established. It was theoretically predicted and experimentally verified that a narrow beam was subject to a self-focusing instability when the input power exceeds a critical threshold, P_c, that depended on the material properties and the wavelength of the light [7,8]. That instability, even at TW power levels, leads to an extreme amplification of the local intensity that is eventually arrested by various physical mechanisms including ionization with subsequent generation of a defocusing plasma [9] and/or dispersion primarily in condensed media. In addition to global collapse of a narrow beam at the critical power, wide beams can experience multiple self-focusing events transverse to the beam. These events are driven by a transverse beam instability that grows from random intensity fluctuations across the beam [10]. So the general picture of high intensity ultrashort pulse propagation is one of a series of randomly generated, self-focusing events across the transverse beam profile.

The simplest canonical mathematical model that includes both self-focusing collapse and the transverse filamentation instability is the scalar Nonlinear Schrödinger equation (NLSE) [7,11]. For this equation the transverse modulational instability and self-focusing collapse are amenable to theoretical analysis and many of the analytic formulas and physical insights that exist in this field are

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based on this mathematical model and its close cousins. The most obvious shortcoming of NLSE itself is that it does not include physical effects that will arrest the collapse. In its purest form it is a equation that forms infinite intensities in finite time [12,13]. Consequently there has been a lot of effort put into coupling this equation to material degrees of freedom that act to regularize the collapse. Coupled NLSE-type models that include both states of polarization have been studied and the self-focusing collapse is not arrested in these models but the collapse threshold is modified. The inclusion of counter propagating modes at this level also fails to arrest the collapse [14].

The effect of an electric field component in the propagation direction, a longitudinal field component, has been less studied but some results are known [15,16]. Formally, a bidirectional ultrashort pulse propagator was derived directly from Maxwell's equation by Kolesik et al. [17] but this involved an expansion in transverse plane wave modes. The unidirectional version of this propagator has been widely used to study ultrashort intense field ionization in gases and condensed media. The influence of longitudinal field components is the focus of the present work. We show that in addition to the longitudinal field components associated with travelling waves which in the paraxial approximation are transverse waves, there is another contribution to the longitudinal field component that we believe has not been investigated so far. This mode is undamped and travels orthogonal to the beam axis. Here we will call it the longitudinal mode. Its motion is controlled by a wave equation in the plane transverse to the beam axis. This wave equation is driven by modes traveling along the beam axis. The size of the driving source is determined by high intensity, short pulse length and nonparaxiality. These effects typically go together in many experimental situations but it is worth noting that the size of the source only requires at least one of them to be present. The size of the source will in its turn determine the strength of the longitudinal mode. The presence of the longitudinal mode can evidently influence the evolution of the self-focusing event by transporting energy away from the collapsing region. It could thus compete with other regularizing effects that have been proposed, like plasma defocusing and higher order Kerr effects [18]. Additionally, the experimental detection of this mode could open a new and previously unexplored window into extreme nonlinear optical interactions of both TW and PW pulses.

The presence of the longitudinal mode is a direct consequence of the Maxwell equations, so its existence is not in question. The effect it might have on the collapse and the practical possibility of detecting it experimentally is an altogether different question that can only be resolved through large scale computation and actual experiments. In this paper we do not address these experimental and computational issues directly but include in Section 2 a detailed description of the equations that must be solved in order to make predictions for specific experimental situations. These equations constitute a reformulation and generalization of the UPPE [19] model referred to above. We use these equations as a starting point for developing asymptotic equations describing the effects of the longitudinal mode in the weakly nonlinear, small bandwidth limit. We also note that our approach leads in a natural way to formulas that can be used to test the accuracy of the UPPE approximation at least in some cases.

In Section 3 we derive the driven wave equation that controls the longitudinal mode. It appears through a solvability condition on the reduced beam equations where only the electric field appears.

In Section 4 we consider the weakly nonlinear, small bandwidth solutions to the evolution equations for the mode amplitudes. We show that in this limit the effect of the longitudinal mode is to limit the range of unstable wavenumbers for the transverse instability and to act as a defocusing lens in the collapse region.

2. The beam equations in the spectral domain

Our basic model equations in this paper are the Maxwell equation including an instantaneous Kerr nonlinearity and linear dispersion:

$$\mathbf{P}(t,x,y,z) = \mathbf{P}_L(t,x,y,z) + \mathbf{P}_{NL}(t,x,y,z)$$
$$\mathbf{P}_L(t,x,y,z) = \varepsilon_0 \int_{-\infty}^{t} dt'\,\chi(t-t')\mathbf{E}(t',x,y,z)$$
$$\mathbf{P}_{NL}(t,x,y,z) = \varepsilon_0 \eta (\mathbf{E}^2\mathbf{E})(t,x,y,z).$$

We introduce the usual beam propagation geometry with the z-axis pointing in the propagation direction. For what we will do it is more convenient to write the Maxwell equations in terms of components. Introducing field and polarization components through

$$\mathbf{E}(t,\mathbf{x}) = e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k}$$
$$\mathbf{B}(t,\mathbf{x}) = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$
$$\mathbf{P}(t,\mathbf{x}) = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$

the Maxwell equations take the form

$$\partial_y e_3 - \partial_z e_2 + \partial_t b_1 = 0$$
$$\partial_z e_1 - \partial_x e_3 + \partial_t b_2 = 0$$
$$\partial_x e_2 - \partial_y e_1 + \partial_t b_3 = 0$$
$$\partial_y b_3 - \partial_z b_2 - \frac{1}{c^2}\partial_t e_1 = \mu_0 \partial_t p_1$$
$$\partial_z b_1 - \partial_x b_3 - \frac{1}{c^2}\partial_t e_2 = \mu_0 \partial_t p_2$$
$$\partial_x b_2 - \partial_y b_1 - \frac{1}{c^2}\partial_t e_3 = \mu_0 \partial_t p_3$$
$$\partial_x b_1 + \partial_y b_2 + \partial_z b_3 = 0$$
$$\varepsilon_0(\partial_x e_1 + \partial_y e_2 + \partial_z e_3) = -\partial_x p_1 - \partial_y p_2 - \partial_z p_3.$$

The beam formulation consists of writing the Maxwell equations as a dynamical system in the z-coordinate. Collecting all of the equations that contain a derivative with respect to z we get

$$\partial_z e_1 = \partial_x e_3 - \partial_t b_2 \tag{1}$$
$$\partial_z e_2 = \partial_y e_3 + \partial_t b_1$$
$$\partial_z e_3 = -\nabla_\perp \cdot e_\perp - \eta(1+L)^{-1}[\nabla_\perp \cdot \left(\left(e_\perp^2 + e_3^2\right) e_\perp\right) + \partial_z \left(e_\perp^2\right) e_3 + \left(e_\perp^2 + 3e_3^2\right)\partial_z e_3]$$
$$\partial_z b_1 = \partial_x b_3 + \frac{1}{c^2}\partial_t\left[\left(1 + L + \eta\left(e_\perp^2 + e_3^2\right)\right)e_2\right]$$
$$\partial_z b_2 = \partial_y b_3 - \frac{1}{c^2}\partial_t\left[\left(1 + L + \eta\left(e_\perp^2 + e_3^2\right)\right)e_1\right]$$
$$\partial_z b_3 = -\partial_x b_1 - \partial_y b_2$$

where we have defined

$$L = \int_{-\infty}^{t} dt'\,\chi(t-t')$$
$$e_\perp = e_1\mathbf{i} + e_2\mathbf{j}.$$

This is not quite a dynamical system because among other things the equation for the longitudinal electric field, e_3, contains derivatives with respect to z on both sides of the equation. We will resolve this problem shortly. For now, observe that the Maxwell system consists of eight equations and we have only used six of them to derive our dynamical system. The remaining two equations give the following two constraints

$$\partial_x e_2 - \partial_y e_1 + \partial_t b_3 = 0$$
$$\partial_x b_2 - \partial_y b_1 - \frac{1}{c^2}\partial_t\left[\left(1 + L + \eta\left(e_\perp^2 + e_3^2\right)\right)e_3\right] = 0 \tag{2}$$

on solutions of the dynamical system (1).Observe however that by using the dynamical system we have

$$\begin{aligned}
&\partial_z\left(\partial_x e_2-\partial_y e_1+\partial_t b_3\right)\\
&=\partial_x\partial_z e_2-\partial_y\partial_z e_1+\partial_t\partial_z b_3\\
&=\partial_x\left(\partial_y e_3+\partial_t b_1\right)-\partial_y\left(\partial_x e_3-\partial_t b_2\right)+\partial_t\left(-\partial_x b_1-\partial_y b_2\right)\\
&=\partial_{xy}e_3+\partial_{xt}b_1-\partial_{yx}e_3+\partial_{yt}b_2-\partial_{tx}b_1-\partial_{ty}b_2\\
&=0
\end{aligned}$$

and

$$\begin{aligned}
&\partial_z\left(\partial_x b_2-\partial_y b_1-\frac{1}{c^2}\partial_t\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_3\right]\right)\\
&=\partial_x\partial_z b_2-\partial_y\partial_z b_1-\frac{1}{c^2}\partial_{zt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_3\right]\\
&=\partial_x\left(\partial_y b_3-\frac{1}{c^2}\partial_t\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_1\right]\right)\\
&\quad-\partial_y\left(\partial_x b_3+\frac{1}{c^2}\partial_t\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_2\right]\right)\\
&\quad-\frac{1}{c^2}\partial_{zt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_3\right]\\
&=\partial_{xy}b_3-\partial_{yx}b_3-\mu_0\partial_t\nabla\cdot\left(\varepsilon_0\boldsymbol{E}+\boldsymbol{P}\right)\\
&=0.
\end{aligned}$$

The constraints are thus conserved by the dynamical system and therefore only need to be imposed at one point along the beam axis. We let this point be located at the point z_0 where the beam enters the medium. The dynamical system (1) is therefore subject to the entry point constraint

$$\begin{aligned}
&\left(\partial_x e_2-\partial_y e_1+\partial_t b_3\right)|_{z_0}=0\\
&\left.\left(\partial_x b_2-\partial_y b_1-\frac{1}{c^2}\partial_t\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_3\right]\right)\right|_{z_0}=0.
\end{aligned}\tag{3}$$

These constraints will turn out to be important for our work. Observe that so far no approximations have been made and the dynamical system (1) with the constraint (3) is equivalent to the original Maxwell equations.

The magnetic field can be eliminated by taking time derivatives of the equations for b_1 and b_2 in (1). This leads to the following beam equations for the electric field components alone

$$\begin{aligned}
\partial_{zz}e_1&=\partial_{xz}e_3+\partial_{xy}e_2-\partial_{yy}e_1\\
&\quad+\frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_1\right]\\
\partial_{zz}e_2&=\partial_{yz}e_3+\partial_{xy}e_1-\partial_{xx}e_2\\
&\quad+\frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_2\right]\\
\partial_z e_3&=-\nabla_\perp\cdot e_\perp-\eta(1+L)^{-1}[\nabla_\perp\cdot((e_\perp^2+e_3^2)e_\perp)\\
&\quad+\partial_z(e_\perp^2)e_3+(e_\perp^2+3e_3^2)\partial_z e_3].
\end{aligned}\tag{4}$$

As propagation equations for the z coordinate this system is somewhat awkward since it is an implicit system for $\partial_z e_3$. We could resolve this by solving the third equation exactly with respect to $\partial_z e_3$. This would produce a somewhat messy explicit equation for $\partial_z e_3$. However observe that since the size of the nonlinear term, as measured by the parameter η, is in all cases of interest a small perturbation, we can solve the third equation in (4) with respect to $\partial_z e_3$ by iterating it once. This gives the system

$$\begin{aligned}
\partial_{zz}e_1&=\partial_{xz}e_3+\partial_{xy}e_2-\partial_{yy}e_1\\
&\quad+\frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_1\right]\\
\partial_{zz}e_2&=\partial_{yz}e_3+\partial_{xy}e_1-\partial_{xx}e_2\\
&\quad+\frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_2\right]\\
\partial_z e_3&=-\nabla_\perp\cdot e_\perp-\eta(1+L)^{-1}[\nabla_\perp\cdot((e_\perp^2+e_3^2)e_\perp)\\
&\quad+\partial_z(e_\perp^2)e_3-(e_\perp^2+3e_3^2)\nabla_\perp\cdot e_\perp]
\end{aligned}$$

but we have

$$\begin{aligned}
&\nabla_\perp\cdot\left(\left(e_\perp^2+e_3^2\right)e_\perp\right)-\left(e_\perp^2+3e_3^2\right)\nabla_\perp\cdot e_\perp\\
&\quad=\nabla_\perp\left(e_\perp^2+e_3^2\right)e_\perp+\left(e_\perp^2+e_3^2\right)\nabla_\perp\cdot e_\perp\\
&\qquad-\left(e_\perp^2+3e_3^2\right)\nabla_\perp\cdot e_\perp\\
&\quad=\nabla_\perp\left(e_\perp^2+e_3^2\right)e_\perp-2e_3^2\nabla_\perp\cdot e_\perp.
\end{aligned}$$

So the electric field equations can be written as

$$\begin{aligned}
\partial_{zz}e_1&=\partial_{xz}e_3+\partial_{xy}e_2-\partial_{yy}e_1\\
&\quad+\frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_1\right]\\
\partial_{zz}e_2&=\partial_{yz}e_3+\partial_{xy}e_1-\partial_{xx}e_2\\
&\quad+\frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta\left(e_\perp^2+e_3^2\right)\right)e_2\right]\\
\partial_z e_3&=-\nabla_\perp\cdot e_\perp-\eta(1+L)^{-1}\left[\nabla_\perp\left(e_\perp^2+e_3^2\right)\cdot e_\perp\right.\\
&\quad\left.-2e_3^2\nabla_\perp\cdot e_\perp+\partial_z\left(e_\perp^2\right)e_3\right].
\end{aligned}\tag{5}$$

We will now rewrite this system in the spectral domain. From this formulation the UPPE approximation is easily described. The key step here is to find all the modes of the linearized system

$$\begin{aligned}
&\partial_{zz}e_1=\partial_{xz}e_3+\partial_{xy}e_2-\partial_{yy}e_1+\frac{1}{c^2}\partial_{tt}\left[(1+L)\,e_1\right]\\
&\partial_{zz}e_2=\partial_{yz}e_3+\partial_{xy}e_1-\partial_{xx}e_2+\frac{1}{c^2}\partial_{tt}\left[(1+L)\,e_2\right]\\
&\partial_z e_3=-\nabla_\perp\cdot e_\perp.
\end{aligned}$$

Assuming that the medium is homogeneous we can use the Fourier transform in space and time and get the following algebraic system for the transformed variables

$$\begin{aligned}
&-k^2\widehat{e_1}=-\xi_1 k\widehat{e_3}-\xi_1\xi_2\widehat{e_2}+\xi_2^2\widehat{e_1}-\frac{1}{c^2}\omega^2 n^2(\omega)\widehat{e_1}\\
&-k^2\widehat{e_2}=-\xi_2 k\widehat{e_3}-\xi_1\xi_2\widehat{e_1}+\xi_1^2\widehat{e_2}-\frac{1}{c^2}\omega^2 n^2(\omega)\widehat{e_2}\\
&ik\widehat{e_3}=-i\xi_1\widehat{e_1}-i\xi_2\widehat{e_2}
\end{aligned}$$

where k is the longitudinal wavenumber and $\xi=(\xi_1,\xi_2)$ is the transverse wavenumber. This algebraic system is easy to solve and we get the following three types of vector modes.

- Right traveling modes

$$\begin{bmatrix}\beta\\0\\-\xi_1\end{bmatrix}e^{i\beta(\omega,\xi)z}e^{i(\xi\cdot\mathbf{x}-\omega t)},\qquad\begin{bmatrix}0\\\beta\\-\xi_2\end{bmatrix}e^{i\beta(\omega,\xi)\mathbf{z}}e^{i(\xi\cdot\mathbf{x}-\omega t)}.$$

- Left traveling modes

$$\begin{bmatrix}\beta\\0\\\xi_1\end{bmatrix}e^{-i\beta(\omega,\xi)z}e^{i(\xi\cdot\mathbf{x}-\omega t)},\qquad\begin{bmatrix}0\\\beta\\\xi_2\end{bmatrix}e^{-i\beta(\omega,\xi)\mathbf{z}}e^{i(\xi\cdot\mathbf{x}-\omega t)}.$$

- Transversely traveling modes

$$\begin{bmatrix}0\\0\\1\end{bmatrix}e^{i(\xi\cdot\mathbf{x}-\omega t)}.$$

We have here defined

$$\mathbf{x}=(x,y)\tag{6}$$

$$\beta(\omega,\xi)=\sqrt{\frac{\omega^2}{c^2}n^2(\omega)-\xi^2}.\tag{7}$$

The classification of the modes into left and right traveling ones of course only holds for positive frequencies. Observe that the transversely traveling mode is purely longitudinal. This is by definition the longitudinal mode we discussed in the introduction. Also observe that in order to get actual traveling modes and to ensure the completeness of the set of modes we assume that the refractive index, $n(\omega)$, is real. In order for the Kramer–Kronig relations to be satisfied $n(\omega)$ is of course in reality complex, but assuming that we are far from any material resonances the imaginary part will be small. The imaginary part can then, if needed, be included with the nonlinearity.

Because of completeness any vector function, in particular any solution to the beam Eq. (5), can be expanded in terms of the modes of the linearized system

$$\mathbf{e}(z,t,\mathbf{x}) = \frac{1}{8\pi^3}\int_0^\infty d\omega \int d\xi \Bigg\{ A_+(z,\omega,\xi)\begin{bmatrix}\beta\\0\\-\xi_1\end{bmatrix} e^{i\beta(\omega,\xi)z} + A_-(z,\omega,\xi)\begin{bmatrix}\beta\\0\\\xi_1\end{bmatrix} e^{-i\beta(\omega,\xi)z} \times B_+(z,\omega,\xi)\begin{bmatrix}0\\\beta\\-\xi_2\end{bmatrix} e^{i\beta(\omega,\xi)z} + B_-(z,\omega,\xi)\begin{bmatrix}0\\\beta\\\xi_2\end{bmatrix} e^{-i\beta(\omega,\xi)z} Q(z,\omega,\xi)\begin{bmatrix}0\\0\\1\end{bmatrix}\Bigg\} e^{i(\xi\cdot\mathbf{x}-\omega t)} + (*)$$

where A_+ and B_+ are amplitudes for right traveling waves of the two states of linear polarization and A_- and B_- are the amplitudes for the left traveling ones. Q is the amplitude for the longitudinal mode.

There is however redundancy in this mode description. Since $\mathbf{e}$ is real we only need 3 half-frequency range quantities to fix it uniquely for each value of z. We have 5 half-frequency range amplitudes. We can thus impose two more constraints without restricting physical solutions to the beam equations. We will use this redundancy to simplify our equations.

We only need quantities defined for positive frequencies but as usual it is more convenient to work with quantities defined for both positive and negative frequencies. This gives us the following representation of the real electric field

$$\mathbf{e}(z,t,\mathbf{x}) = \frac{1}{8\pi^3}\int d\omega \int d\xi \Bigg\{ A_+(z,\omega,\xi)\begin{bmatrix}\beta\\0\\-\xi_1\end{bmatrix} e^{i\beta(\omega,\xi)z} + A_-(z,\omega,\xi)\begin{bmatrix}\beta\\0\\\xi_1\end{bmatrix} e^{-i\beta(\omega,\xi)z} \times B_+(z,\omega,\xi)\begin{bmatrix}0\\\beta\\-\xi_2\end{bmatrix} e^{i\beta(\omega,\xi)z} + B_-(z,\omega,\xi)\begin{bmatrix}0\\\beta\\\xi_2\end{bmatrix} e^{-i\beta(\omega,\xi)z} \times Q(z,\omega,\xi)\begin{bmatrix}0\\0\\1\end{bmatrix}\Bigg\} e^{i(\xi\cdot\mathbf{x}-\omega t)}.$$

This description is even more redundant than the previous one and we have the usual additional constraints on the amplitudes that ensure the reality of the electric field

$$\begin{aligned} A_-(z,\omega,\xi) &= A_+^*(z,-\omega,-\xi) \\ B_-(z,\omega,\xi) &= B_+^*(z,-\omega,-\xi) \\ Q(z,\omega,\xi) &= Q^*(z,-\omega,-\xi). \end{aligned} \quad (8)$$

When we use this representation the mode amplitudes A_- and B_- are redundant and will in fact not appear in the final equation. In this representation the positive spectral content of A_+ and B_+ are amplitudes for right moving waves and their negative spectral content are amplitudes for left moving waves. The real refractive index function $n(\omega)$ is extended as an even function to negative frequencies.

The component form of the mode expansion is

$$\begin{aligned} e_1(z,t,\mathbf{x}) &= \frac{1}{8\pi^3}\int d\omega d\xi \{\beta A_+ e^{i\beta z} + \beta A_- e^{-i\beta z}\} e^{i(\xi\cdot\mathbf{x}-\omega t)} \\ e_2(z,t,\mathbf{x}) &= \frac{1}{8\pi^3}\int d\omega d\xi \{\beta B_+ e^{i\beta z} + \beta B_- e^{-i\beta z}\} e^{i(\xi\cdot\mathbf{x}-\omega t)} \\ e_3(z,t,\mathbf{x}) &= \frac{1}{8\pi^3}\int d\omega d\xi \{-\xi_1 A_+ e^{i\beta z} - \xi_2 B_+ e^{i\beta z} \\ &\quad \times \xi_1 A_- e^{-i\beta z} + \xi_2 B_- e^{-i\beta z} + Q\} e^{i(\xi\cdot\mathbf{x}-\omega t)}. \end{aligned} \quad (9)$$

Observe that the longitudinal electric field contains both contributions from A_+, B_+ and Q. In a paraxial situation the contribution from A_+ and B_+ is negligible whereas the contribution from Q can still be significant.

We are now going to insert these expressions into the beam equation (5). First we compute the derivatives with respect to z of e_1 and e_2:

$$\begin{aligned} \partial_z e_1(z,t,\mathbf{x}) &= \frac{1}{8\pi^3}\int d\omega \int d\xi \{i\beta^2 A_+ e^{i\beta z} + \beta\partial_z A_+ e^{i\beta z} \\ &\quad - i\beta^2 A_- e^{-i\beta z} + \beta\partial_z A_- e^{-i\beta z}\} e^{i(\xi\cdot\mathbf{x}-\omega t)} \\ \partial_z e_2(z,t,\mathbf{x}) &= \frac{1}{8\pi^3}\int d\omega \int d\xi \{i\beta^2 B_+ e^{i\beta z} + \beta\partial_z B_+ e^{i\beta z} \\ &\quad - i\beta^2 B_- e^{-i\beta z} + \beta\partial_z B_- e^{-i\beta z}\} e^{i(\xi\cdot\mathbf{x}-\omega t)}. \end{aligned}$$

We can simplify these expressions considerably by imposing the constraints [20]

$$\begin{aligned} \partial_z A_+ e^{i\beta z} + \partial_z A_- e^{-i\beta z} &= 0 \\ \partial_z B_+ e^{i\beta z} + \partial_z B_- e^{-i\beta z} &= 0. \end{aligned} \quad (10)$$

What is happening here is completely analogous to what happens when we apply the method of variation of parameters in order to find special solutions to second order ordinary differential equations. In that case we introduce two unknown functions, the parameters, but we have only one equation to satisfy. We can thus specify one more relation between the parameters and this is done in such a manner that the equations for the parameters become simplified. Integrating the constraint (10) between the boundary of the domain, z_0 and some arbitrary point z we get the identity

$$A_-(z,\omega,\xi) = A_-(z_0,\omega,\xi) - \int_{z_0}^{z} dz' \partial_{z'} A_+(z',\omega,\xi) e^{2i\beta z'}$$

where $A_-(z_0,\omega,\xi)$ can be given any value. This means that the constraint does not determine the left propagating amplitude in terms of the right propagating one. We still have full freedom to fit any given electric field configurations at z_0. Note however the remark after Eq. (14).

The constraints (10) now imply that

$$\partial_z e_1(z,t,\mathbf{x}) = \frac{1}{8\pi^3}\int d\omega d\xi \{i\beta^2 A_+ e^{i\beta z} - i\beta^2 A_- e^{-i\beta z}\} e^{i(\xi\cdot\mathbf{x}-\omega t)} \quad (11)$$

$$\partial_z e_2(z,t,\mathbf{x}) = \frac{1}{8\pi^3}\int d\omega d\xi \{i\beta^2 B_+ e^{i\beta z} - i\beta^2 B_- e^{-i\beta z}\} e^{i(\xi\cdot\mathbf{x}-\omega t)}$$

$$\partial_z e_3(z,t,\mathbf{x}) = \frac{1}{8\pi^3}\int d\omega d\xi \{-i\xi_1\beta A_+ e^{i\beta z} - i\xi_2\beta B_+ e^{i\beta z} - i\xi_1\beta A_- e^{-i\beta z} - i\xi_2\beta B_- e^{-i\beta z} - 2\xi_1\partial_z A_+ e^{i\beta z} - 2\xi_2\partial_z B_+ e^{i\beta z} + \partial_z Q\} e^{i(\xi\cdot\mathbf{x}-\omega t)}.$$

We now take another z derivative of the expressions for e_1 and e_2. Observe that because of the constraints (10) the expressions for $\partial_{zz}e_1$ and $\partial_{zz}e_2$ contain only first derivatives of the amplitudes with respect to z.

Inserting these expressions for e_1, e_2 and their derivatives into the beam Eq. (5) gives after some manipulations the system

$$\{2i(\beta^2+\xi_1^2)\partial_z A_+ + 2i\xi_1\xi_2\partial_z B_+\}e^{i\beta z} - i\xi_1\partial_z Q = \widehat{NL_1}$$
$$\{2i(\beta^2+\xi_2^2)\partial_z B_+ + 2i\xi_1\xi_2\partial_z A_+\}e^{i\beta z} - i\xi_2\partial_z Q = \widehat{NL_2}$$
$$\{-2\xi_1\partial_z A_+ - 2\xi_2\partial_z B_+\}e^{i\beta z} + \partial_z Q = \widehat{NL_3}$$

for the amplitudes. Here the hat symbol, $\widehat{\ }$ above any quantity means the Fourier transform in x and t. The nonlinear terms NL_1, NL_2 and NL_3 are defined by

$$NL_1 = \frac{\eta}{c^2}\partial_{tt}\left[\left(\left(e_\perp^2+e_3^2\right)\right)e_1\right]$$
$$NL_2 = \frac{\eta}{c^2}\partial_{tt}\left[\left(\left(e_\perp^2+e_3^2\right)\right)e_2\right]$$
$$NL_3 = -\eta(1+L)^{-1}\left[\nabla_\perp\left(e_\perp^2+e_3^2\right)\cdot e_\perp - 2e_3^2\nabla_\perp\cdot e_\perp + \partial_z\left(e_\perp^2\right)e_3\right].$$

By addition and subtraction we can simplify the linear part of this system and arrive at the following system of first order equations for the spectral amplitudes

$$2i\beta\partial_z A + 2\beta^2 A = \widehat{NL_1} + i\xi_1\widehat{NL_3} \quad (12)$$
$$2i\beta\partial_z B + 2\beta^2 B = \widehat{NL_2} + i\xi_2\widehat{NL_3}$$
$$\partial_z Q = \frac{\omega^2 n^2(\omega)}{c^2\beta^2}\widehat{NL_3} - i\frac{\xi_1}{\beta^2}\widehat{NL_1} - i\frac{\xi_2}{\beta^2}\widehat{NL_2}.$$

This system is equivalent to the full Maxwell equations under the assumption that the nonlinear terms are smaller than the linear terms. The negative frequency content of the amplitudes A and B will describe the left traveling waves and the positive frequency content will describe the right traveling waves. Q is the amplitude for the longitudinal mode. In order to generate a solution consisting of both left and right traveling waves we must specify the full spectral content at z_0. Observe that (12) appears to be implicit in $\partial_z A$ and $\partial_z B$ since NL_3 contains $\partial_z e_1$ and $\partial_z e_2$. However since we are imposing the constraints (10) these derivatives only depend on A and B and not on $\partial_z A$ and $\partial_z B$ as one might have believed. Thus (12) is an explicit set of propagation equations for the z coordinate.

In the UPPE approximation we assume that the negative frequency content of the amplitudes at $z = z_0$ is exactly zero and any negative frequency content that might be generated during the numerical propagation of the amplitudes is removed. Evidently this is not a consistent solution to the system (12) since the nonlinearity will drive negative frequency content up from noise that is always present. The success of the UPPE approximation depends on whether or not the negative frequency content that is generated by the actual system is small compared to the positive frequency content. This can be verified *a posteriori* using the constraints (10) and (8). By integration we get the relations

$$A_+^*(z,-\omega,-\xi) = A_+^*(z_0,-\omega,-\xi) - \int_{z_0}^{z} dz'\partial_z A_+(z',\omega,\xi)e^{2i\beta z'} \quad (13)$$

$$B_+^*(z,-\omega,-\xi) = B_+^*(z_0,-\omega,-\xi) - \int_{z_0}^{z} dz'\partial_z B_+(z',\omega,\xi)e^{2i\beta z'}.$$

In the typical experimental setup there is no source at $z = +\infty$. Thus

$$A_+^*(z,-\omega,-\xi) = A_-(z,\omega,\xi) \longrightarrow 0$$
$$B_+^*(z,-\omega,-\xi) = B_-(z,\omega,\xi) \longrightarrow 0$$

when $z \longrightarrow +\infty$. Using these limiting values the identities (13) turn into

$$A_+^*(z_0,-\omega,-\xi) = \int_{z_0}^{\infty} dz'\partial_z A_+(z',\omega,\xi)e^{2i\beta z'} \quad (14)$$

$$B_+^*(z_0,-\omega,-\xi) = \int_{z_0}^{\infty} dz'\partial_z B_+(z',\omega,\xi)e^{2i\beta z'}.$$

We can now compute the solutions using the UPPE approximation and then compute the integrals occurring in (14). If the calculated values for $A_+^*(z_0,-\omega,-\xi)$ and $B_+^*(z_0,-\omega,-\xi)$ are small compared to $A_+(z_0,\omega,\xi)$ and $B_+(z_0,\omega,\xi)$ we can be assured that the UPPE approximation is accurate. If they are not small we can use the identities (14) as the basis for an iterative scheme where the negative frequency content at $z = z_0$ is repeatedly updated until the procedure converges to a fixed negative spectral content at z_0. Note that Eq. (14) does not contradict our statements made after Eq. (10) since the assumption that there is no source at infinity should and will constrain the possible amplitudes at z_0.

3. The wave equation for the purely longitudinal mode

We will in this section show that the amplitude for the purely longitudinal mode satisfies a certain wave equation in a plane transverse to the beam axis. The traveling mode amplitudes act as a source in this wave equation. The strength of the source is determined by high intensity, short pulse length and nonparaxiality.

In order to clarify the logic of our argument let us first consider a simpler situation where there is only one transverse dimension. The electric and magnetic fields are given by the components

$$\mathbf{E}(z,x,t) = e\mathbf{j}$$
$$\mathbf{B}(z,x,t) = b_1\mathbf{i} + b_2\mathbf{k}.$$

The dynamical system for this case is

$$\partial_z e = \partial_t b_1 \quad (15)$$
$$\partial_z b_1 = \partial_x b_2 + \frac{1}{c^2}\partial_t\left[\left(1+L+\eta e^2\right)e\right]$$
$$\partial_z b_2 = -\partial_x b_1$$

with a constraint preserved by the dynamical system

$$\partial_x e + \partial_t b_2 = 0. \quad (16)$$

The corresponding closed system for the electric field alone is

$$\partial_{zz} e = -\partial_{xx} e + \frac{1}{c^2}\partial_{tt}\left[\left(1+L+\eta e^2\right)e\right]. \quad (17)$$

The question we ask is whether or not all solutions to (17) are physical in the sense that for any solution e there exists an electromagnetic field satisfying Maxwell's equations and that has $e\mathbf{j}$ as an electric field component. Since the constrained dynamical system (15) is equivalent to Maxwell's equations the question is if for any solution of (17) we can find functions b_1 and b_2 such that the dynamical system (15) and the constraint (16) are both satisfied.

Any solution to (17) is determined uniquely by the specification of both e and $\partial_z e$. The specification of $\partial_z e$ turns the first equation in (15) into a constraint on the magnetic field component b_1. Thus taking into account the constraint (16) we have the following system of equations that has to be satisfied by the magnetic field components

$$\partial_t b_1 = \partial_z e$$
$$\partial_t b_2 = -\partial_x e.$$

We have a determinate system of two equations for the two unknown magnetic field components that can be solved for all choices of e and $\partial_z e$. Thus we get the very unsurprising conclusions that all solutions to (17) are physical.

When we apply the same logic to the beam Eqs. (5) and the corresponding dynamical system (1) we get an overdetermined system of equations for the magnetic field components. This overdetermined system leads to a wave equation for the purely longitudinal mode.

Any solution to the system (5) is uniquely determined by a specification of $e_1, e_2, e_3, \partial_z e_1$ and $\partial_z e_2$. The specification of $\partial_z e_1$ and $\partial_z e_2$ turns the first two equations in (1) into two constraints on the magnetic field components. Taking into account the constraints (2) we have the following system of equations that has to be satisfied by the magnetic field components

$$\partial_t b_1 = \partial_z e_2 - \partial_y e_3$$
$$\partial_t b_2 = \partial_x e_3 - \partial_z e_1$$
$$\partial_t b_3 = \partial_y e_1 - \partial_x e_2$$
$$\partial_x b_2 - \partial_y b_1 = \frac{1}{c^2}\partial_t\left[\left(1 + L + \eta\left(e_\perp^2 + e_3^2\right)\right)e_3\right].$$

In this case we have four equations for the three magnetic field components. The system is thus overdetermined and can only be solved if the right hand side satisfies the constraint

$$\nabla_\perp^2 e_3 - \frac{1}{c^2}\partial_{tt}(1+L)e_3 = \partial_{xz}e_1 + \partial_{yz}e_2 + \frac{\eta}{c^2}\partial_{tt}((e_\perp^2 + e_3^2)e_3).$$

If we now insert the mode expansions (9) and apply the constraint (10) cancellations occur and we end up with the equation

$$\nabla_\perp^2 q - \frac{1}{c^2}\partial_{tt}\,(1+L)\,q = \partial_{tt}N \tag{18}$$

where

$$q(z,t,\mathbf{x}) = \frac{1}{8\pi^3}\int d\omega d\xi\, Q(z,\omega,\xi)e^{i(\xi\cdot\mathbf{x}-\omega t)}$$
$$N = \frac{\eta}{c^2}((e_\perp^2 + e_3^2)e_3).$$

Note that if we assume that N is the only source of q, in the absence of N we should use the solution $q = 0$. In general the source will depend on q so the wave equation is nonlinear. From (18) it is evident that high intensity and short pulses will lead to a large source and to a large contribution to the longitudinal field from q. Taking the Fourier transform of (18) we get in the spectral domain

$$Q(z,\omega,\xi) = -\frac{\omega^2}{\beta^2}\widehat{N}(z,\omega,\xi)$$

where

$$\beta^2 = \left(\left(\frac{\omega}{c}\right)^2 n^2(\omega) - \xi^2\right).$$

If the fields are highly nonparaxial, $\widehat{N}(z,\omega,\xi)$ will have nonzero contributions in the spectral space (ω,ξ) where β is close to zero. This will lead to an enhancement of the mode amplitude Q.

Thus high intensity,short pulses and nonparaxiality all separately contribute to a source generating the mode Q that radiates transversely away from the collapse region. These pulse properties often go together in physical situations but we only need one of them to create a significant source for Q.

4. Weakly nonlinear envelope approximation

The purely longitudinal mode will remove energy from the collapse region so it will evidently have a stabilizing effect. The extent and specific characteristics of this stabilizing effect will be found through large scale simulations of the evolution equations for the mode amplitudes (12). However considering the equations in the weakly nonlinear, paraxial and small bandwidth limit can give some analytic insight into what will happen in the more general setting. We will therefore consider this situation now.

We will in the following assume that the amplitude for the y-polarized mode, B, is set to zero. This is not really a consistent solution to the Eqs. (12) since any small amount of noise in the B mode will be driven up by the nonlinear terms. However in the simplified analysis we present here this possibility will be disregarded; its presence or not will not change the nature of the result we will present in this section.

Therefore the mode equations that we will solve approximately are

$$2i\beta\partial_z A + 2\beta^2 A = \widehat{NL_1} + i\xi_1\widehat{NL_3} \tag{19}$$
$$\partial_z Q = \frac{\omega^2 n^2(\omega)}{c^2\beta^2}\widehat{NL_3} - i\frac{\xi_1}{\beta^2}\widehat{NL_1} - i\frac{\xi_2}{\beta^2}\widehat{NL_2}.$$

Observe that due to the constraints (8) we can write the mode expansion for the remaining electric field components in the form

$$e_1(z,t,\mathbf{x})$$
$$= \frac{1}{8\pi^3}\int d\omega d\xi\{\beta A_+ e^{i\beta z} + \beta A_- e^{-i\beta z}\}e^{i(\xi\cdot\mathbf{x}-\omega t)} \tag{20}$$
$$= \frac{1}{8\pi^3}\sum_s \int d\omega d\xi A^s e^{is(\xi\cdot\mathbf{x}-\omega t)} \tag{21}$$
$$e_3(z,t,\mathbf{x})$$
$$= \frac{1}{8\pi^3}\int d\omega d\xi\{-\xi_1 A_+ e^{i\beta z} + \xi_1 A_- e^{-i\beta z} + Q\}e^{i(\xi\cdot\mathbf{x}-\omega t)}$$
$$= \frac{1}{8\pi^3}\sum_s \int d\omega d\xi \frac{1}{\beta}\left\{-s\xi_1 A^s + \frac{1}{2}Q^s\right\}e^{is(\xi\cdot\mathbf{x}-\omega t)} \tag{22}$$

where s is a binary index with values $+$and $-$. We have here defined for A

$$A^s = \begin{cases} A & \text{if } s = + \\ A^* & \text{if } s = - \end{cases}$$

and the same for Q^s.

We use the multiple scale approach [21] assuming a leading order for the amplitudes of the form

$$A(z,\omega,\xi) = \frac{1}{\varepsilon^2}A_0\left(z, \frac{\omega-\omega_0}{\varepsilon}, \frac{\xi}{\varepsilon}\right) + \cdots$$
$$Q(z,\omega,\xi) = \frac{1}{\varepsilon^2}Q_0\left(z, \frac{\omega-\omega_0}{\varepsilon}, \frac{\xi}{\varepsilon}\right) + \cdots.$$

For the electric field components e_1 and e_3 this implies that to leading order

$$e_1(z,t,\mathbf{x})$$
$$= \frac{1}{8\pi^3}\sum_s \int d\omega d\xi A^s e^{is(\xi\cdot\mathbf{x}-\omega t)}$$
$$\approx \frac{1}{8\pi^3}\sum_s \int d\omega d\xi \frac{1}{\varepsilon^2}A_0^s\left(z, \frac{\omega-\omega_0}{\varepsilon}, \frac{\xi}{\varepsilon}\right)e^{is(\xi\cdot\mathbf{x}-\omega t)}$$
$$= \frac{\varepsilon}{8\pi^3}\sum_s \int d\Omega d\mathbf{K} A_0^s(z,\Omega,\mathbf{K})e^{is\mathbf{K}\cdot(\varepsilon\mathbf{x})}e^{is\Omega(\varepsilon t)}e^{-is\omega_0 t}$$
$$= \sum_s A_0^s(z,\varepsilon t,\varepsilon\xi)e^{-is\omega_0 t} + \mathcal{O}(\varepsilon^2) \tag{23}$$

and

$$
\begin{aligned}
&e_3(z,t,\mathbf{x}) \\
&= \frac{1}{8\pi^3}\sum_s \int d\omega d\xi \frac{1}{\beta}\left\{-s\xi_1 A^s + \frac{1}{2}Q^s\right\} e^{is(\xi\cdot\mathbf{x}-\omega t)} \\
&\approx \frac{1}{8\pi^3}\sum_s \int d\omega d\xi \frac{1}{\beta}\left\{-\frac{s\xi_1}{\varepsilon^2}A_0^s\left(z,\frac{\omega-\omega_0}{\varepsilon},\frac{\xi}{\varepsilon}\right)\right. \\
&\quad \left. + \frac{1}{2\varepsilon^2}Q_0^s\left(z,\frac{\omega-\omega_0}{\varepsilon},\frac{\xi}{\varepsilon}\right)\right\} e^{is(\xi\cdot\mathbf{x}-\omega t)} \\
&= \frac{\varepsilon}{8\pi^3}\sum_s \int \frac{1}{\beta}\left\{-s\varepsilon\Omega_1 A_0^s(z,\Omega,\mathbf{K}) + \frac{1}{2}Q_0^s(z,\Omega,\mathbf{K})\right\} \\
&\quad \times e^{is\mathbf{K}\cdot(\varepsilon\mathbf{x})}e^{is\Omega(\varepsilon t)}e^{-is\omega_0 t} \\
&= \sum_s Q_0^s(z,\varepsilon t,\varepsilon\xi)e^{-is\omega_0 t} + \mathcal{O}(\varepsilon^2) \qquad (24)
\end{aligned}
$$

where

$$
\begin{aligned}
&A_0(z,\varepsilon t,\varepsilon\xi) \\
&= \frac{\varepsilon}{8\pi^3}\int d\Omega d\mathbf{K} A_0(z,\Omega,\mathbf{K})e^{i\mathbf{K}\cdot(\varepsilon\mathbf{x})}e^{i\Omega(\varepsilon t)} \\
&Q_0(z,\varepsilon t,\varepsilon\xi) \\
&= \frac{\varepsilon}{16\beta_{000}^0\pi^3}\int d\Omega d\mathbf{K} Q_0(z,\Omega,\mathbf{K})e^{i\mathbf{K}\cdot(\varepsilon\mathbf{x})}e^{i\Omega(\varepsilon t)}
\end{aligned}
$$

and where $\beta_{000}^0 = \beta(\omega_0, 0, 0)$.

Thus A_0 and Q_0 are of order ε and have variation of order one on time scales and space scales of order $1/\varepsilon$. We use a multiple scale approach to find consistent approximations to Eqs. (19). The application of the multi-scale method is well known and the details will not be discussed in this paper. Here we will only make some remarks on the size on the various nonlinear terms. In the multiple scale approach we assume that

$$
\begin{aligned}
A_0 &= \sum_v \varepsilon^v A_{0v} \qquad (25)\\
Q_0 &= \sum_v \varepsilon^v Q_{0v}
\end{aligned}
$$

where the A_{0v} and Q_{0v} are functions of multiple z scales $z_j = \varepsilon^j z$ with $j \geq 0$. The evolution of the leading order amplitudes A_{00} and Q_{00} on the slow scales z_j for $j \geq 1$ are determined so as to render the expansions (25) uniform on progressively longer length scales.

From Eqs. (19) we get for the fastest z scale the system

$$
\begin{aligned}
&2i\beta_{000}^0\partial_{z_0}A_{00} + 2(\beta_{000}^0)^2 A_{00} = 0 \\
&\partial_z Q_{00} = 0
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
&A_{00}(z_0,z_1,\ldots,\Omega,\mathbf{K}) = B_{00}(z_1,\ldots,\Omega,\mathbf{K})e^{i\beta_{000}^0 z_0} \\
&Q_{00}(z_0,z_1,\ldots,\Omega,\mathbf{K}) = C_{00}(z_1,\ldots,\Omega,\mathbf{K}).
\end{aligned}
$$

From (23) and (24) it is evident that

$$NL_1 = \mathcal{O}(\varepsilon^3).$$

For the terms of NL_3 we have the orders

$$
\begin{aligned}
&\nabla_\perp\left(e_\perp^2 + e_3^2\right)\cdot e_\perp = \mathcal{O}(\varepsilon^4) \\
&2e_3^2\nabla_\perp\cdot e_\perp = \mathcal{O}(\varepsilon^4) \\
&\partial_z\left(e_\perp^2\right)e_3 = \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Thus overall the nonlinearity NL_3 is of order ε^3. However when we use the multiple scale method, only the terms in NL_1 that have the same variation at the fastest scale z_0 as A_{00} will contribute to the evolution of A_{00} on the slower scales $z_1, z_2, \ldots$. The effects of the other terms will average to zero on the slower scales. For NL_3 the same logic applies and the only terms that will contribute to the evolution of Q_{00} on the slower scales $z_1, z_2, \ldots$ are terms that are constant on the fastest scale z_0. Therefore the only terms from $\partial_z\left(e_\perp^2\right)e_3$ that will contribute are terms in $e_\perp^2$ where the fast oscillation $e^{i\beta_{000}^0 z_0}$ cancels. But then ∂_z of those terms is not of order one but of order ε. So the terms from $\partial_z\left(e_\perp^2\right)e_3$ that actually contribute to the slow evolution of Q_{00} are all of order ε^4. The second equation in (19) then shows that Q will vary on scales $z_4 = \varepsilon^4 z$ and higher. This tells us that in order to get a consistent approximation to the system (19) we must apply the multiple scale method at least up to order ε^4.

The nonlinearities of the system (19) are composed of NL_1 and NL_3 and applying the same kind of logic as above we are lead to the simplified system

$$
\begin{aligned}
&2i\beta\partial_z A + 2\beta^2 A = \widehat{NL_1} \qquad (26)\\
&\partial_z Q = \frac{\omega^2 n^2(\omega)}{c^2\beta^2}\widehat{NL_3}
\end{aligned}
$$

where we have removed all terms that cannot contribute to the slow evolution.

Applying the multiple scale method to this system gives us the following amplitude equations:

$$
\begin{aligned}
\partial_z\mathcal{A} \;&=\; -\beta_{100}^0\partial_t\mathcal{A} - i\beta_{200}^0\partial_{tt}\mathcal{A} - i\beta_{020}^0\nabla_\perp^2\mathcal{A} \\
&\quad + \beta_{102}^0\partial_t\nabla_\perp^2\mathcal{A} + \beta_{300}^0\partial_{ttt}\mathcal{A} \\
&\quad + i\alpha_1\left(6|\mathcal{A}|^2 + |\mathcal{Q}|^2\right)\mathcal{A} - \alpha_2\partial_t\left[\left(6|\mathcal{A}|^2 + |\mathcal{Q}|^2\right)\mathcal{A}\right] \qquad (27)\\
\partial_z\mathcal{Q} &= -\alpha_3\partial_z|\mathcal{A}|\mathcal{Q}
\end{aligned}
$$

where α_1, α_2 and α_3 are positive real constants and where

$$\beta_{ijk}^0 = \frac{1}{i!j!k!}\partial_{\omega\xi_1\xi_2}^{i+j+k}\beta\Big|_{\omega=\omega_0,\xi_1=0,\xi_2=0}$$

are the Taylor coefficients of the propagation constant.

4.1. Transverse modulational instability

It is well known that the culprit behind the small scale filamentation of intense laser pulses is the transverse modulational instability. In order to see the effect of the longitudinal mode on transverse modulational instability we consider the simplest possible situation where we have only diffraction and nonlinearity. Including all terms in the amplitude equations change the details of the conclusions presented in this section but not their essence. Thus we consider the system

$$
\begin{aligned}
&\partial_z\mathcal{A} = -\beta_{100}^0\partial_t\mathcal{A} - i\beta_{020}^0\nabla_\perp^2\mathcal{A} + i\alpha_1\left(6|\mathcal{A}|^2 + |\mathcal{Q}|^2\right)\mathcal{A} \qquad (28)\\
&\partial_z\mathcal{Q} = -\alpha_3\partial_z|\mathcal{A}|\mathcal{Q}.
\end{aligned}
$$

The system has exact solutions of the form

$$
\begin{aligned}
&\mathcal{A} = \mathcal{A}_0 e^{ik_0 z}e^{i(\xi_0\cdot\mathbf{x}-\omega_0 t)} \\
&\mathcal{Q} = \mathcal{Q}_0
\end{aligned}
$$

where

$$k_0 = \beta_{100}^0\omega_0 + \beta_{020}^0\xi^2 + \alpha_1(6|\mathcal{A}_0|^2 + |\mathcal{Q}_0|^2).$$

These solutions represent an infinitely broad laser beam.

We now introduce a small perturbation on these exact solutions

$$
\begin{aligned}
&\mathcal{A} = (1+a)\mathcal{A}e^{ik_0 z}e^{i(\xi_0\cdot\mathbf{x}-\omega_0 t)} \\
&\mathcal{Q} = (1+c)\mathcal{Q}_0.
\end{aligned}
$$

Inserting these solutions into Eqs. (28) and retaining only terms of first order in a and q we get the system

$$\begin{aligned}\partial_z a &= -\beta^0_{100}\partial_t a + 2\beta^0_{020}\xi_0\cdot\nabla_\perp a - i\beta^0_{020}\nabla^2_\perp a\\ &\quad + i\alpha_1\left(6|\mathcal{A}_0|^2(a+a^*) + |\mathcal{Q}_0|^2(c+c^*)\right)\\ \partial_z a^* &= -\beta^0_{100}\partial_t a^* + 2\beta^0_{020}\xi_0\cdot\nabla_\perp a^* + i\beta^0_{020}\nabla^2_\perp a^*\\ &\quad - i\alpha_1\left(6|\mathcal{A}_0|^2(a+a^*) + |\mathcal{Q}_0|^2(c+c^*)\right)\end{aligned}$$

$$\partial_z c = -\alpha_3|\mathcal{A}_0|^2(c+c^*)$$
$$\partial_z c^* = -\alpha_3|\mathcal{A}_0|^2(c+c^*).$$

Taking the Fourier transform of this system we get

$$\begin{aligned}ka_+ &= \beta^0_{100}\omega a_+ + 2\beta^0_{020}\xi_0\cdot\xi a_+ + \beta^0_{020}\xi^2 a_+\\ &\quad + \alpha_1\left(6|\mathcal{A}_0|^2(a_+ + a_-) + |\mathcal{Q}_0|^2(c_+ + c_-)\right)\\ ka_- &= \beta^0_{100}\omega a_- + 2\beta^0_{020}\xi_0\cdot\xi a_- - \beta^0_{020}\xi^2 a_-\\ &\quad - \alpha_1\left(6|\mathcal{A}_0|^2(a_+ + a_-) + |\mathcal{Q}_0|^2(c_+ + c_-)\right)\end{aligned}$$

$$c_+ = -\alpha_3|\mathcal{A}_0|^2\partial_z(c_+ + c_-)$$
$$c_- = -\alpha_3|\mathcal{A}_0|^2\partial_z(c_+ + c_-)$$

where for simplicity we have defined

$$a_+ = \hat{a}(k,\omega,\xi)$$
$$a_- = \hat{a}^*(-k,-\omega,-\xi).$$

The system can be written more simply as

$$\begin{aligned}pa_+ &= \beta^0_{020}\xi^2 a_+ + q(a_+ + a_-)\\ pa_- &= -\beta^0_{020}\xi^2 a_- - q(a_+ + a_-)\end{aligned} \tag{29}$$

where

$$p = k - \beta^0_{100}\omega - 2\beta^0_{020}\xi_0\cdot\xi$$
$$q = 2\alpha_1|\mathcal{A}_0|^2(3 - 2\alpha_3|\mathcal{Q}_0|^2).$$

The system (29) has a nontrivial solution only if

$$p^2 = (\beta^0_{020})^2\xi^4 + 2q\beta^0_{020}\xi^2$$

and from the definition of the propagation constant (6) we observe that

$$\beta^0_{020} = -\frac{1}{\beta^0_{000}}$$

so we can write

$$p^2 = \left(\frac{1}{\beta^0_{000}}\right)^2\xi^2(\xi^2 - 2q\beta^0_{000}).$$

From this it is evident that a perturbation with transverse wavenumber ξ will experience exponential growth along the beam axis if

$$\xi^2 < 2q\beta^0_{000} = r|\mathcal{A}_0|^2(3 - 2\alpha_3|\mathcal{Q}_0|^2)$$

where $r = 2\alpha_1\beta^0_{000}$. This formula show clearly that the effect of the longitudinal mode in this approximation is to restrict the range of wavenumbers that experience transverse modulational instability.

4.2. The defocusing lens property

The self-focusing collapse of a intense laser pulse happens because the refractive index is intensity dependent

$$n(I) = n_0 + n_2 I.$$

Such an index acts as a focusing lens that increases the intensity which in its turn focuses more strongly and so on without end in the ideal case. We will now see that the presence of the longitudinal mode adds a term to $n(I)$ that acts as a defocusing lens. What happens will then be a competition between these two lensing effects. Using as in the previous subsection the simplest version of the amplitude equations we have

$$\begin{aligned}\partial_z\mathcal{A} &= -\beta^0_{100}\partial_t\mathcal{A} - i\beta^0_{020}\nabla^2_\perp\mathcal{A} + i\alpha_1\left(6|\mathcal{A}|^2 + |\mathcal{Q}|^2\right)\mathcal{A}\\ \partial_z\mathcal{Q} &= -\alpha_3\partial_z|\mathcal{A}|\mathcal{Q}.\end{aligned} \tag{30}$$

Observe that the last equation can be solved explicitly. In fact we have

$$\partial_z\mathcal{Q} = -\alpha_3\partial_z|\mathcal{A}|\mathcal{Q}$$
$$\Downarrow$$
$$\partial_z|\mathcal{Q}|^2 = -2\alpha_3\partial_z|\mathcal{A}|\,|\mathcal{Q}|^2$$
$$\Downarrow$$
$$|\mathcal{Q}|^2 = De^{-\alpha_3|\mathcal{A}|}$$

where D in general can be any function of the transverse coordinates and time. The amplitude equation for $\mathcal{A}$ now becomes

$$\partial_z\mathcal{A} = -\beta^0_{100}\partial_t\mathcal{A} - i\beta^0_{020}\nabla^2_\perp\mathcal{A} + i\alpha_1\left(6|\mathcal{A}|^2 + De^{-\alpha_3|\mathcal{A}|}\right)\mathcal{A}.$$

This corresponds to the existence of a intensity dependent refractive index of the form

$$n(I) = n_0 + n_2 I + n_3 e^{-2\alpha_3\sqrt{I}},$$

where n_3 is some new parameter that can depend on the transverse coordinates. Since the intensity is higher close to the beam axis than far away from the beam axis during a collapse, the last term is smaller on the beam axis than further off the beam axis. This means that, by itself, it will act as a defocusing lens in the same way that the term $n_2 I$, by itself, acts as a focusing lens. Taken together the system is still focusing; the effect of the Q mode is to make it somewhat less focusing.

The last two subsections show that in the limiting case we have been discussing the effect of the longitudinal mode is to make the system less prone to filamentation instability and self-focusing collapse.

5. Summary

We have in this paper investigated the effect of the existence of a longitudinally polarized mode traveling transversely to the beam axis. We have seen that in the weakly nonlinear envelope case the mode tends to make the beam propagation less prone to filamentation and self-focusing collapse. We have furthermore shown that in the general case the mode obeys a dispersive wave equation in the transverse coordinates that is driven by the amplitudes for the modes traveling along the beam axis and also by the mode itself in general. The longitudinal mode will thus transport energy away from the collapse and therefore in general the mode will tend to make the system less prone to collapse. The mode can also give a new insight into the collapse dynamics if it can be detected experimentally. The existence of the mode and the existence of the effects discussed in this work is not in question, but whether the effect is of a size that makes it significant for the evolution of the collapse and filamentation processes and experimentally detectable is a question that can only be decided by large scale numerical simulations and actual experiments.

However for the numerical solution of (12) we are faced with a calculation involving time and three space dimensions. Adding to this the presence of a very extensive support in spectral space, the numerical calculations appear daunting. There is however a simplified situation involving TM fields in a slab geometry that have many of the same features as the full system discussed in this paper, including the transversely propagating longitudinal mode. This system is much more amenable to numerical simulations and results from these simulations will be reported in a forthcoming publication.

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