This text is a brief recap of the basic properties of spatial solitons. In OPTI-583, we look at both spatial and temporal solitons — the mathematics of the latter is in one-to-one correspondence with what is summarized next.

0.0.1 Physical backround for spatial optical solitons

The equation to describe spatial soliton beam propagation is:

$$\frac{\partial \mathcal{E}}{\partial z} = \frac{i}{2k_0}\Delta_\perp \mathcal{E} + \frac{i\omega}{2cn(\omega)}\mathcal{P}/\epsilon_0 = \frac{i}{2k_0}\Delta_\perp \mathcal{E} + ik_0 n_2 I_{unit}|\mathcal{E}|^2\mathcal{E}$$

It represents a beam propagating in one transverse dimension, and subject to self-focusing nonlinearity due to optical Kerr effect. Single transverse dimension is an idealization, of course. In reality, the beam would be confined to a planar waveguide, and $n(\omega)$ would be replaced by the effective index of the fundamental mode — effective index approximation then results in the above propagation equation. This dimensional reduction requires approximations; in particular one has to assume that higher-order modes of the planar wave-guide are not excited. If this is in fact a good approximation very much depends on the conditions, but even in high-contrast waveguides where higher order modes are a possibility, the light propagates in the fundamental mode (if properly launched, of course) also in the presence of nonlinear interactions. For the moment we accept this model as given.

Specifically for one transverse dimension, solutions exists in which the self-focusing effects are balanced by diffraction. This balance is perfectly steady for the so-called fundamental soliton, and it is "dynamic" for higher-order (which means higher energy or power in the beam) solutions. Higher order solitons have the same spatial profile with the fundamental one at periodic propagation distances. This is very convenient for our purposes, because it allows us to initialize higher-order nonlinear solutions without explicit use of analytic formulas.

Before going further, note that the balance between the forces of diffraction and nonlinear focusing, even if dynamic as in the case of higher-order solitons, is only made possible by the one-dimensional character of the diffraction, and the fact that the intensity in the shrinking beam increases slower (as a function of its transverse dimension) than it would in two (transverse) dimensions.

Fundamental solution

Solution for the fundamental soliton can be obtained easily from the following Ansatz

$$\mathcal{E} = A_0 \frac{1}{\cosh(x/w)} \exp(i\beta z) \ .$$

Insert this into the propagation equation, and cancel some nonzero common terms that occur on both sides, to get

$$\beta = \frac{1}{2w^2 k_0} - \frac{1}{k_0 w^2}\frac{1}{\cosh^2(x/w)} + k_0 n_2 I_{unit} A_0^2 \frac{1}{\cosh^2(x/w)} \ .$$

Once again, this is nothing but a solvability condition that determines the propagation constant and thus chromatic dispersion properties of the soliton. For this to hold as an identity, both β, and A_0 must be fixed such that the propagation constant equals the first term on the right, and the two x-dependent terms cancel each other. The latter condition relates the intensity to the spatial width of the soliton:

$$k_0^2 n_2 I_{unit} A_0^2 w^2 = 1 \ .$$

Since we have two parameters, A_0 and w, only constrained by one relation, there is a whole family of fundamental solitons. The wider is the beam, the lower intensity is sufficient to hold it together and compensate the diffraction.

This is a special case of fundamental (spatial) soliton. It provides a convenient way to test numerical implementation of the pulse propagator. One question is whether numerics can "preserve" an initial condition which coincides with this exact solution. This is what we will look at.

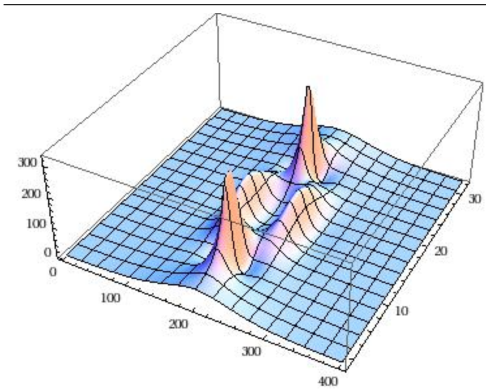
Another question is what happens if the initial beam has a bell shape but is not exactly the same as the fundamental soliton. It can be shown that the soliton solution is rather robust: It will emerge from the initial condition which will shed the excessive energy (beam power) and adjust its shape to $\text{sech}(x/w)$. HOwever, for the simulation to be able to handle such a dynamics, transparent boundary conditions must be used. The radiation shed by the to-be soliton must be able to disappear through the computational domain edge, leaving behind a properly formed soliton.

Higher-order solutions

Higher-order solitons can be initialized in a very similar way, because at certain distances, namely when $\beta z = \pi/4$, they assume the same spatial shape as the fundamental one. We can simply start simulation from this special position (thus renaming the origin of the z axis). However, one must adjust the relation between the beam width w and its initial intensity A_0^2 to

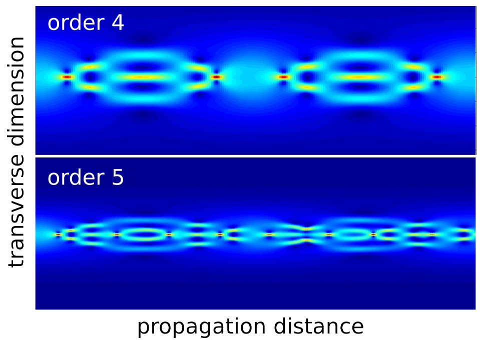
$$k_0^2 n_2 I_{unit} A_0^2 w^2 = N^2 \ .$$

Here N stands for the soliton number. If it happens to be an integer, a periodically repeating solution will appear. This is yet another opportunity to put a nonlinear-pulse simulatir to a good-strength test: Even without knowing the explicit functional form of the higher-order soliton, one can easily tell if the numerical solution does what it should simply by inspecting if it creates a nice periodic-in-z intensity pattern. The following is an example of the intensity profile in a third-order spatial soliton:



Spatial profile of the intensity in a third-order spatial soliton. With the initial cross-section is identical to that of the fundamental soliton, the evolution along the propagation distance is shown over a single period. The periodicity of a numerical solution offers a convenient test — inaccuracies in the integration of the evolution equation accumulate until a departure from the strict periodicity becomes obvious after a few soliton periods. This trend is more obvious in higher-order solitons.

Of course, higher order soliton solutions are more difficult to simulate. Since they have more complex profiles and richer dynamics, they require both finer spatial resolution and shorter integration step. As a result, the computational effort needed to obtain even a couple of periods of a higher-order soliton with a reasonable quality may be several times bigger than that required for a lower-order soliton. The following figure illustrates these issues:



Numerical simulation of higher-order solitons. The upper panel shows the evolution of the intensity in the fourth-order solution. The result clearly shows the periodic structure, and only has minor deviations from the strict periodicity (visible as a slight asymmetry in the middle of the panel). The lower panel shows an attempt to simulate the soliton of order five. While the numerical solution exhibits the characteristic features with multiple symmetric intensity peaks across the transverse dimension, it is clearly not periodic along z. Higher resolution and a finer integration step would be necessary to correct the problem.

Propagation at angle

If one adds a linear phase-shift to the initial condition spatial profile, the outcome of the evolution is still a soliton, but one that propagates at an angle proportional to its phase-front tilt. This is expressed through the formula

$$\mathcal{E} = A_0 \frac{1}{\cosh\left(\frac{x - z k_x/k_0}{w}\right)} \exp\left(i k_x x\right) \exp\left(i \beta z\right) \quad , \quad \beta = \frac{1}{2k_0 w^2} - \frac{k_x^2}{2k_0} \ ,$$

where the first term describes the envelope moving with a constant transverse "velocity," and the last expression show the change in the propagation constant.

Simulations of the above waveform can be done in the spatial domain that utilizes (in gUPPEcore) the one-dimensional AXIS_LINEAR which has natural periodic boundary conditions. However, close-to-soliton initial conditions will not converge to the true soliton because in such a "periodic box" there is no way to shed the excess energy...