In[8]:=
```
P[a_, b_] := (1 + a x + I b x) / (1 + a x - I b x)
```

In[206]:=
```
Clear[a4, a1, a2, a3, b]
```

In[207]:=
```
prop = P[a2, dz b] P[a1 , dz b]
```

Out[207]=
$$\frac{(1 + a1\ x + i\ b\ dz\ x)\ (1 + a2\ x + i\ b\ dz\ x)}{(1 + a1\ x - i\ b\ dz\ x)\ (1 + a2\ x - i\ b\ dz\ x)}$$

In[208]:=
```
target = Exp[ I dz (Sqrt[1 + x] - 1)]
```

Out[208]=
$$e^{i\ dz\ \left(-1+\sqrt{1+x}\right)}$$

In[209]:=
```
constr = Collect[Simplify[Normal[Series[target - prop, {x, 0, 4}]]], x]
```

Out[209]=
$$\frac{1}{384} i\ (192 - 1536\ b)\ dz\ x + \frac{1}{384} i\ dz\ \left(768\ a1\ b + 768\ a2\ b - 3072\ i\ b^2\ dz + 48\ i\ (i + dz)\right) x^2 + \frac{1}{384} i\ dz$$
$$\left(-768\ a1^2\ b - 768\ a2^2\ b + 3072\ i\ a1\ b^2\ dz + 3072\ i\ a2\ b^2\ dz + 4608\ b^3\ dz^2 - 8\ \left(-3 + 3\ i\ dz + dz^2\right)\right) x^3 +$$
$$\frac{1}{384} i\ dz\ \left(-15 + 768\ a1^3\ b + 768\ a2^3\ b + 15\ i\ dz - 3840\ i\ a1^2\ b^2\ dz - 1536\ i\ a1\ a2\ b^2\ dz -\right.$$
$$\left. 3840\ i\ a2^2\ b^2\ dz + 6\ dz^2 - 6912\ a1\ b^3\ dz^2 - 6912\ a2\ b^3\ dz^2 - i\ dz^3 + 6144\ i\ b^4\ dz^3\right) x^4$$

In[210]:=
```
eqns = Table[Coefficient[constr, x^l], {l, 1, 4}]
```

Out[210]=
$$\left\{\frac{1}{384} i\ (192 - 1536\ b)\ dz, \frac{1}{384} i\ dz\ \left(768\ a1\ b + 768\ a2\ b - 3072\ i\ b^2\ dz + 48\ i\ (i + dz)\right), \frac{1}{384} i\ dz\right.$$
$$\left(-768\ a1^2\ b - 768\ a2^2\ b + 3072\ i\ a1\ b^2\ dz + 3072\ i\ a2\ b^2\ dz + 4608\ b^3\ dz^2 - 8\ \left(-3 + 3\ i\ dz + dz^2\right)\right),$$
$$\frac{1}{384} i\ dz\ \left(-15 + 768\ a1^3\ b + 768\ a2^3\ b + 15\ i\ dz - 3840\ i\ a1^2\ b^2\ dz - 1536\ i\ a1\ a2\ b^2\ dz -\right.$$
$$\left.\left. 3840\ i\ a2^2\ b^2\ dz + 6\ dz^2 - 6912\ a1\ b^3\ dz^2 - 6912\ a2\ b^3\ dz^2 - i\ dz^3 + 6144\ i\ b^4\ dz^3\right)\right\}$$

In[211]:=
```
solb = Solve[eqns[[1]] == 0, b][[1]]
```

Out[211]=
$$\left\{b \to \frac{1}{8}\right\}$$

In[212]:=
```
eqns = Simplify[Table[Coefficient[constr /. solb, x^l], {l, 1, 4}]]
```

Out[212]=
$$\left\{0, \frac{1}{8} i\ (-1 + 2\ a1 + 2\ a2)\ dz, \frac{1}{384} dz\ \left(24\ i - 96\ i\ a1^2 - 96\ i\ a2^2 + 24\ dz - 48\ a1\ dz - 48\ a2\ dz + i\ dz^2\right),\right.$$
$$-\frac{1}{768} dz\ \left(30\ i - 192\ i\ a1^3 - 192\ i\ a2^3 + 30\ dz - 120\ a1^2\ dz -\right.$$
$$\left.\left. 120\ a2^2\ dz + 3\ a1\ \left(-16\ a2 + 9\ i\ dz\right)\ dz - 12\ i\ dz^2 + 27\ i\ a2\ dz^2 + dz^3\right)\right\}$$

In[216]:=
```
Simplify[eqns /. {a1 → 1 / 4 - s, a2 → 1 / 4 + s}]
```

Out[216]=
$$\left\{0, 0, \frac{1}{384} i\ dz\ \left(12 + dz^2 - 192\ s^2\right), -\frac{dz\ \left(48\ i + 24\ dz + 3\ i\ dz^2 + 2\ dz^3 - 576\ i\ s^2 - 384\ dz\ s^2\right)}{1536}\right\}$$

In[217]:=
```
sols = Solve[%[[3]] == 0, s][[1]]
```

Out[217]=
$$\left\{s \to -\frac{\sqrt{12 + dz^2}}{8\sqrt{3}}\right\}$$

In[218]:=
```
sola = {a1 → 1 / 4 - s , a2 → 1 / 4 + s} /. sols
```

Out[218]=
$$\left\{a1 \to \frac{1}{4} + \frac{\sqrt{12 + dz^2}}{8\sqrt{3}}, a2 \to \frac{1}{4} - \frac{\sqrt{12 + dz^2}}{8\sqrt{3}}\right\}$$

In[219]:=
```
propapr = prop /. sola /. solb
```

Out[219]=
$$\frac{\left(1+\frac{i\ dz\ x}{8}+\left(\frac{1}{4}-\frac{\sqrt{12+dz^2}}{8\sqrt{3}}\right)x\right)\left(1+\frac{i\ dz\ x}{8}+\left(\frac{1}{4}+\frac{\sqrt{12+dz^2}}{8\sqrt{3}}\right)x\right)}{\left(1-\frac{i\ dz\ x}{8}+\left(\frac{1}{4}-\frac{\sqrt{12+dz^2}}{8\sqrt{3}}\right)x\right)\left(1-\frac{i\ dz\ x}{8}+\left(\frac{1}{4}+\frac{\sqrt{12+dz^2}}{8\sqrt{3}}\right)x\right)}$$

In[223]:=
```
Series[target - propapr, {x, 0, 5}]
```

Out[223]=
$$-\frac{1}{128}\,i\,dz\,x^4+\frac{i\left(270\,dz-90\,i\,dz^2+15\,dz^3+dz^5\right)x^5}{23040}+O[x]^6$$

In[221]:=
```
sola
```

Out[221]=
$$\left\{a1\to\frac{1}{4}+\frac{\sqrt{12+dz^2}}{8\sqrt{3}},\ a2\to\frac{1}{4}-\frac{\sqrt{12+dz^2}}{8\sqrt{3}}\right\}$$

In[222]:=
```
solb
```

Out[222]=
$$\left\{b\to\frac{1}{8}\right\}$$