Purpose:

- Modified radial C-N method applied to the 2D self-focusing collapse. Implementation with the extrapolated nonlinear term contributing to the right-hand-side of the C-N system.
- Self-focusing collapse produces a beam solution that is violently contracting in diameter and increases in intensity as it approaches singularity (a point at which the solution ceases to exist). Availability of an accurate formula predicting the propagation distance needed to reach self-focusing collapse offers yet another opportunity for testing the nonlinear beam propagation code. Attempts to resolve a collapsing solution should motivate the implementation of adaptive control for the integration step.

Beam propagation in self-focusing medium

The following provides a qualitative argument for the existence of critical power for self-focusing collapse for beam propagation in two transverse dimensions (i.e. in free space). It will be shown that no matter what is the beam diameter, if its power exceeds the so-called critical power for self-focusing (denoted P_c), self-focusing will overcome diffraction. As a result, the beam will shrink and eventually collapse in a finite distance. This is in a sharp contrast to the existence of soliton solutions we explored in one transverse dimension.

Note: What follows is a scaling argument, in which factors of the order of unity are not essential.

Consider beam with power P and diameter D

Index "inside" the beam:

$$n_i = 1 + \Delta n = 1 + n_2 I = 1 + n_2 \frac{P}{D^2}$$

Index "outside" the beam:

$$n_o = 1$$

Rays with small angle will be totally reflected on the boundary between "inside" and "outside." The critical angle of propagation α_c must obey

$$n_i \cos\alpha_c = n_o \cos(0) = 1 \ .$$

Since the critical angle is small, cosine can be Taylor expanded, so that we have

$$(1 + n_2 \frac{P}{D^2})(1 - \frac{\alpha_c^2}{2}) = 1 \ .$$

Neglecting small quantities "of second order", leads to

$$n_2 \frac{P}{D^2} = \frac{\alpha_c^2}{2} \ .$$

Diffraction will prevail over self-focusing if "maximal" angle of propagation of a ray from the beam is larger than α_c. To estimate the maximal angle of propagation that is contained in the beam, one has to combine the only two scales such an estimate can depend on, namely λ and D. With these two, the dimensionless ratio λ/D can serve as a rough estimate of angular content of the beam. The condition therefore is

$$\alpha_c < \frac{\lambda}{D}$$

which leads us to

$$n_2 \frac{P}{D^2} = \frac{\alpha_c^2}{2} < \frac{\lambda^2}{2D^2} \quad \text{or} \quad P =< \frac{\lambda^2}{2n_2}$$

Note: This condition is independent of beam size D, which indicates that the beam fate (to diffract or to collapse?) only depends on the total power, and not on intensity.

Note: This situation is qualitatively different from propagation in one transverse dimension. Repeat the above argument for one transverse dimension, and show that diffraction can eventually balance the self-focusing tendency. It is this balance that made the solitons in the previous exercise possible.

Self-focusing collapse: results useful for BPM testing

The refractive index n in the presence of an intense electromagnetic field does not only depend on its frequency, but also on the local intensity $I(r)$ of the laser beam. It is usually characterized in term of the nonlinear index n_2:

$$n = n_0 + n_2 I(r)$$

The coefficient of the nonlinear Kerr index n_2 corresponds to the third-order nonlinear susceptibility. It is usually positive, which results in an increase of the refractive index in the presence of intense radiation. Typical values are $n_2 \approx 10^{-23}\mathrm{m}^2/\mathrm{W}$ for gases, and three orders of magnitude higher for condensed media.

The highest intensity normally occurs in the central part of a beam. That means that also the induced index change is concentrated closer to axis, which in turn acts as an effective lens. We have shown in the previous section that if power is higher than certain threshold, catastrophic collapse must follow.

Because we only gave a qualitative, scaling justification of this, the question is what is the precise value of critical power. It turns out that there is a special beam profile, called Townes, for which an unstable balance occurs, at which the beam neither diffracts nor collapses. The power for which this happens is

$$P_{cr} = 3.72\lambda^2/8\pi n_0 n_2$$

Beams with different profiles exhibit somewhat different critical power values. For example, in the case of a Gaussian beam the coefficient in the above formula has to be changed to 3.77.

While the final behavior of the beam is independent of the diameter, the propagation length before the self-focusing collapse of course increases with the beam size. The distance to collapse is well approximated by the following empirical formula (called Marburger):

$$L_c = \frac{0.367 L_{df}}{\sqrt{[(\frac{P_{in}}{P_{cr}})^{1/2} - 0.852]^2 - 0.0219}}$$

where L_{df} is the Rayleigh range of the beam. This equation is reasonably accurate for Gaussian beams with moderate powers. For higher powers, onset of modulation instability breaks the beam into multiple filaments.

If the beam is initially focused with a lens of focal length f, then the collapse point moves closer to the lens, and the new collapse distance becomes

$$\frac{1}{L_{c,f}} = \frac{1}{L_c} + \frac{1}{f}$$

We will use the above formulas (i.e. Marburger and lens) for testing our implementation of nonlinear BPM with Kerr effect. Students should observe that rather accurate agreement with the above prediction can be achieved with appropriate grid resolution and careful integration.

Of course, because the finite grid spacing, the simulated collapse is recognized only as a sharp increase of intensity. The behavior in the immediate vicinity of the collapse singularity depends and the concrete numerical method and its parameters. In some algorithms, the point of singularity can be overcome and the solution can be extended; The singularity thus manifest as a very sharp peak in the intensity as a function of propagation distance. This is a typical behavior in robust spectral codes. In other methods, numerical solution may crash when approaching the collapse point.