Numerical wave dispersion in two transverse dimensions

Derivation of the C-N dispersion relation for two transverse dimensions is a simple generalization of the one-dimensional procedure outlined in our lecture notes.

The eigenfunctions of the discrete, two-dimensional Laplacian Δ can be parametrized by two transverse wavenumbers k_x and k_y. They must come in the form of plane waves:

$$E^n_{ps} \equiv E(n\Delta z, p\Delta x, s\Delta y) = Ae^{iK_z z_n} e^{ik_x x_p} e^{ik_y y_s} \equiv Ae^{iK_z z_n} \psi_{ps}(k_x, k_y) \ . \tag{1}$$

Here we have to use double indices to denote a specific location on the computational grid. Calculation of the corresponding eigenvalue is straightforward, yielding

$$\begin{aligned} \sum_{ps} \Delta_{mn,ps}\psi_{ps} \quad &= \\ &+ \quad (e^{ik_x x_{m-1}} - 2e^{ik_x x_m} + e^{ik_x x_{m+1}})e^{ik_y y_n} \\ &+ \quad (e^{ik_y y_{n-1}} - 2e^{ik_y y_n} + e^{ik_y y_{n+1}})e^{ik_x x_m} \\ &= \quad [2(\cos k_x \Delta x - 1) + 2(\cos k_y \Delta y - 1)]\psi_{mn}. \end{aligned} \tag{2}$$

As one can expect, this eigenvalue is just a sum of eigenvalues corresponding to two orthogonal directions in the grid.

The corresponding propagation constant is obtained after using the above in the Crank-Nicolson update formula, which leads to

$$K_z(k_x, k_y) = \frac{-i}{\Delta z} \ln \left[\frac{1 + \frac{i\Delta z}{2k_0 \Delta x^2}(\cos k_x \Delta x - 1) + \frac{i\Delta z}{2k_0 \Delta y^2}(\cos k_y \Delta y - 1)}{1 - \frac{i\Delta z}{2k_0 \Delta x^2}(\cos k_y \Delta x - 1) - \frac{i\Delta z}{2k_0 \Delta y^2}(\cos k_y \Delta y - 1)} \right] \ , \tag{3}$$

This is clearly anisotropic, resulting in different propagation properties for different relative values of k_x and k_y (i.e. different direction in the transverse plane). We can see that the nature of anisotropy is similar to what we encountered in the section on finite-difference Maxwell solvers.

So, the question stands, what is the number of grid points needed per transverse wavelength in order to reduce the anisotropy to an acceptable level?

The following figure shows the ratio between the propagation constants of waves propagating with transverse wavevectors along coordinate axis and along the grid diagonal, respectively. Plots correspond to several values of $\delta = \Delta z/(4k_0\Delta x)$, and indicate that large n **and** small Δz must be used to suppress anisotropy. The message here is that for practical purposes **certain degree of anisotropy must be accepted**.