Purpose:

Practical implementation of the discrete Hankel transform in the context of BPM.

Summary of DHT:

What the discrete Hankel transform should do is a counterpart of the continuous formulas for the Hankel transform (here we restrict our attention strictly to the zero-order transforms):

$$F(k) = \int_0^\infty r dr J_0(kr) f(r) \quad , \quad f(r) = \int_0^\infty k dk J_0(kr) F(k) \tag{1}$$

Forward and backward transformation has exactly the same form, and one can verify these formulas using the orthogonality relation for the Bessel functions:

$$\int_0^\infty r dr J_0(kr) J_0(ur) = \frac{1}{k} \delta(k-u) \tag{2}$$

Indeed,

$$F(k) = \int_0^\infty r dr J_0(kr) f(r) = \int_0^\infty r dr J_0(kr) \int_0^\infty u du J_0(ur) F(u) = \int_0^\infty u du F(u) \int_0^\infty r dr J_0(kr) J_0(ur) = \int_0^\infty u du F(u) \frac{\delta(k-u)}{k} = F(k) \tag{3}$$

This forward-backward symmetry is something that can be preserved in the numerical implementation, too.

First, assume that the function $f(r)$ is nonzero only for $r \in (0, R_{max})$, and that also its spatial spectrum $F(k)$ is band-limited, and vanishes for wavenumbers k beyond some maximal spatial frequency. The first assumption allows to restrict the computational domain to a finite (radial) interval starting at $r = 0$ and extending up to R_{max}. The limited bandwidth assumption makes it possible to use a finite number of grid points to capture all necessary spatial frequencies.

The computational grid in the real space is then given by a set of points

$$r_k = R_{max} \frac{\alpha_k}{\alpha_M} \quad , \quad k = 1, 2, \ldots, M-1 \tag{4}$$

where α_k is the k-th zero of J_0, and M determines the grid resolution and therefore the maximal spatial frequency it can support. Note that the very first grid point lies at non-zero distance from the axis $r = 0$. The last point r_{M-1} is also away from the boundary R_{max} at which $f(R_{max}) = 0$; The grid spatial resolution can be roughly characterized by R_{max}/M, but the points are not spaced at equal distances.

The corresponding grid in the spectral space is

$$k_n = \frac{\alpha_n}{R_{max}} \quad , \quad k = 1, 2, \ldots, M-1 \ . \tag{5}$$

The minimal spatial frequency is α_1/R_{max} ($\alpha_1 \approx 2.4048$). This is the counterpart of the transverse wavenumber "quantum" $2\pi/L$ for a linear spatial domain of length L. The maximal frequency supported by the grid is k_{M-1}, and it scales up with the number of points M.

Functions representing optical beam profile amplitudes will be sampled at these points, and notations

$$F_n \equiv F(k_n) \quad f_k \equiv f(r_k) \tag{6}$$

will be used. The discrete version of the Hankel transform is simply matrix multiplication by the same matrix H,

$$F_n = \sum_k H_{nk} f_k \quad , \quad f_k = \sum_n H_{kn} F_n \ . \tag{7}$$

Since the continuous transform is its own inverse, the same is expected of the discrete version, so the square of H should give an identity matrix:

$$\sum_k H_{nk} H_{km} = \delta_{nm} \tag{8}$$

Johnson [1] gives the explicit form for H:

$$H_{mn} = \frac{2}{\alpha_M} \frac{J_0(\alpha_m \alpha_n / \alpha_M)}{J_1^2(\alpha_n)} \tag{9}$$

Reference [1] also provides a proof that (8) holds in the limit of large M: repeated application of H is

$$\sum_k H_{nk} H_{km} = \frac{4}{\alpha_M^2} \sum_{k=1}^{M-1} \frac{J_0(\alpha_n \alpha_k / \alpha_M)}{J_1^2(\alpha_k)} \frac{J_0(\alpha_k \alpha_m / \alpha_M)}{J_1^2(\alpha_m)} = \delta_{nm} \tag{10}$$

which is orthogonality relation (11) in Ref. [1]. For finite M, above is not strictly true, but even for smallest Ms the accuracy is rather high. Deviations will show up as non-zero off-diagonal elements in H^2 which should be an identity matrix if the orthogonality was analytically exact. Typically, few hundred of points are used to populate a radial computational domain, and the corrections are practically lost in the numerical "noise."

[1] H. Fisk Johnson, *Computing discrete Hankel transform*, Computer Physics Communications 43(1987)181.

Task 1: Implement a Matlab script for a DHT function. It should carry a structure holding the transformation matrix, and arrays of coordinates in both the real space and spectral space (i.e. transverse wavenumbers).

<u>Instructor's solution:</u> myDHT.m, plus a testing script testDHT.m, which just creates a DHT object, defines a select mode in both the real space and spectral space, and calculates the corresponding transformed counterparts.

Task 2:

Use your DHT implementation to perform simulation of the Poisson's bright spot similar to that done in the previous work-package. Utilize the program/script previously written for the 2D, FFT-based BPM to execute the same simulation with programs implementing both methods. You should obtain numerical results which agree very well — this will be the first test of the DHT-BPM. Note, that the simulation time required for the DHT approach is just a fraction of that needed for the 2D FFT method.

<u>Instructor's solution:</u> Scripts p1FFT.m and p2DHT.m are be set-up to execute "the same" simulation. Results are written in *amplitude_vs_x_FFT.dat* and *amplitude_vs_radius_DHT.dat*, respectively, and can be used to plot the resulting field amplitudes for detailed comparison.