Purpose: This exercise introduces an elegant method to obtain the complete dispersion relation of the numerical solver, i.e.

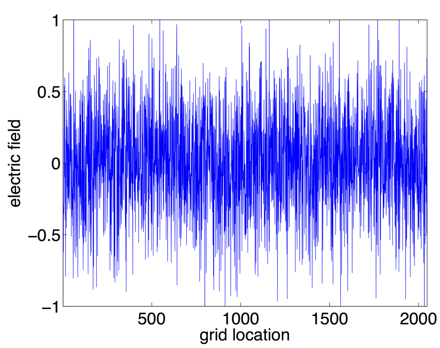
$$\sin\left(\frac{\omega\Delta t}{2}\right)^2 = \left(\frac{\Delta t}{\Delta x}\right)^2 \sin\left(\frac{k\Delta x}{2}\right)^2 \ ,$$

in a single simulation run. The implementation of this method only calls for a very minor modification of the 1D Maxwell solver implemented in the previous assignment.

Method: The idea utilizes the fact that wave propagation is linear. So, one can initialize all possible types of waves and observe their behavior in a single simulation. Surely, such a simulation must contain information about the propagation of waves of arbitrary wavelengths. The following is the algorithm that allows to extract the numerical dispersion relation from such a simulation.

Step 1.
One way to imprint "all" wavelength in the single initial condition supplied to the solver is to create it as white noise (in both electric and magnetic field). Such a chaotic initial field contains short as well as long-wavelength waves that will propagate independently through the lattice.

Step 2.
Starting from the white-noise initial condition, the simulator is run for a fixed (large) number of steps. The following figure shows one snapshot of the evolving electric field:



It is not surprising that field snapshots that are result of evolution from a white noise initial condition all look the same, namely as white noise. Yet, as an ensemble they carry full information about the numerical dispersion relation governing all waves that the simulator can realize.

These electric field snapshots are stored after every integration step. The result is a two-dimensional array (matrix) of

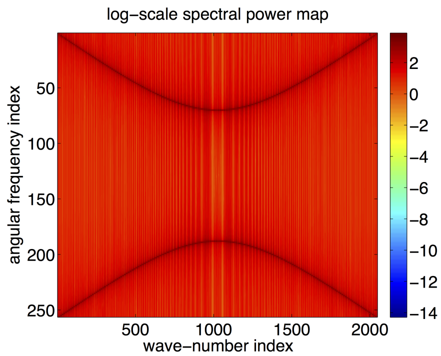
$$M_{i,j} = E(x_i, t_j) \ .$$

(Note that x_i and t_j here stand for discrete "index" values corresponding to the grid nodes and discrete time steps.)

Step 3.
The next step is to calculate two-dimensional Fourier transform of this matrix, to obtain the spectrum

$$\hat{M}_{u,v} = \mathrm{FFT}(M)_{u,v} = \hat{E}(k_u, \omega_v) \ .$$

(Where u and v label discrete values of wave-numbers and angular frequency.) This picture shows the logarithmic-scale map of $\hat{M}$. The lines reveal the numerical dispersion relation.



Two dimensional Fourier spectrum of of the field $E(x,t)$ amplitude that was initiated by white noise. The dark lines, marking loci where the spectral energy concentrates, show the dispersion relation of the numerical method.

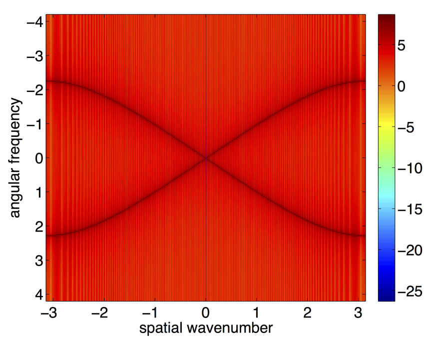
The vertical axis of the above map spans the array of temporal frequencies defined through the Fourier transform of a temporal evolution of the field at each spatial point. With NT steps executed by the solver, the duration of the temporal domain becomes $NT\Delta t$. Consequently the discrete frequencies (sampled by the FFT) are

$$\omega_v = v\frac{2\pi}{NT\Delta t} \qquad v = -\frac{NT}{2},\ldots,+\frac{NT}{2}$$

The horizontal axis index labels the allowed wave-vectors

$$k_u = u\frac{2\pi}{NX\Delta x} \qquad u = -\frac{NX}{2},\ldots,+\frac{NX}{2}$$

In this coordinate system, the spectral power map looks like this (after Matlab's fftshift):



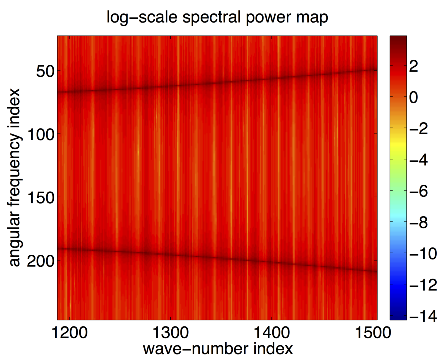
Two dimensional Fourier spectrum of of the field $E(x,t)$ amplitude that was initiated by white noise. Physical coordinates (with $c = 1$ and $\Delta x = 1$ are used in this map. Waves with approximately correct propagation properties are now in the central portion of the figure.

It is now straightforward to verify that the loci of spectral power concentration indeed correspond to the numerical wave dispersion relation of the 1D Maxwell solver. The four nearly linear segments emanating from the center represent numerical waves that approximate their physical counterparts accurately. There are four "branches," because the simulated field is real-valued (and both positive and negative frequencies must be present), and because there are two possible propagation directions. Because the slope of the curve, $\partial\omega/\partial k$ is the group velocity, we can see that numerical waves with the shortest wavelengths (close to edges of the figure) do not propagate at all!

Task A
Explain why and how the above-described method works.

Task B
Explain how the choice of the simulation parameters Δx, Δt, and their ratio affect where in the map does the dispersion relation locus appear. Convert the theoretical dispersion relation into the "coordinate system" of the spectrum-matrix $\hat{M}$, and verify the simulation result.

Task C
Zoom in the above picture reveals a texture, consisting of semi-regular lines:



A zoom into the power spectrum shows a system of "lines." Their spacing seems to exhibit definite periodicity, which indicates that this is no random noise.

In numerical simulation practice it is important not to overlook artifacts — every unexpected feature must be understood: Explain where the lines originate from. Is this effect "mostly harmless," or is it necessary to control it?

Hint: When trying to decide if something (in numerical simulation results) is an artifact, it is often useful to list all "un-physical" parameters that control the simulation. In this case, we have Δx, Δt, their ratio, number of spatial grid points NX, the number of executed steps NT, and the random number generator seed and algorithm. Having identified what quantities have the potential to introduce artificial effects, one can study the response of the system to changes of these parameters. More often than not one can uncover clues that help to explain what is going on...