Fractal structure of eigenmodes of unstable-cavity lasers

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We show that the eigenmodes of unstable-cavity lasers have fractal structure, in contrast with the well-known stable-cavity eigenmodes. As with all fractals, the dynamic range over which self-similarity holds is limited; in this case the range is set by diffraction, i.e., by the Fresnel number of the resonator. We determine the fractal dimension of the mode profiles and show that it is related to the aperture shape. © 1998 Optical Society of America

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The eigenmodes of stable-cavity lasers can easily be calculated analytically to be the well-known Hermite–Gaussian modes; as such they hold no surprises. In contrast, finding the eigenmodes of hard-edged unstable-cavity lasers is far from trivial, and one generally has to resort to numerical techniques, such as virtual-source theory.[1–3] The outcome of such calculations is a rather complicated mode profile with many Fresnel ripples that defies intuitive interpretation. In this Letter we show that these mode profiles have an unexpected underlying structure: We demonstrate that they are self-similar or fractal in nature.

To illustrate our point we discuss as a generic example a one-dimensional hard-edged confocal unstable cavity (Fig. 1). The aperture of this strip resonator is defined by the small outcoupling mirror.

Unstable cavities are characterized by two numbers, the round-trip magnification M and the equivalent Fresnel number N, which for a confocal cavity is given by

$$N = \frac{1}{2}(M - 1)\frac{a^2}{\lambda L}, \tag{1}$$

where a is the mirror radius, λ is the wavelength, and L is the cavity length.[4]

The eigenmodes of such a cavity are fully determined by M and N. We restrict the discussion to the lowest-loss mode. The transverse intensity profile for this mode is shown in Fig. 2 for three values of N, as calculated with virtual-source theory by use of the paraxial approximation.[5]

It is clear that for higher Fresnel numbers more details are present. A physical argument for the size of the smallest details is as follows: A uniformly illuminated lens of radius a and a Fresnel number N give a focal spot size of the order of a/N.[4] We thus expect the smallest details to be separated by $1/N$ on the horizontal scale in Fig. 2, which agrees with the details of the curves.

Our key observation is this: Each round trip leads to a magnified image of the intensity distribution in the aperture plane; this is an essential property of unstable-cavity resonators that is absent in stable-cavity resonators. The eigenmode intensity distribution in the aperture plane does not change after one round trip (otherwise it would not be an eigenmode). But a round trip amounts to magnification, so the eigenmode must be invariant under magnification, and this is exactly the definition of a fractal. As with all physical fractals, this self-similarity cannot hold down to infinitely small scales; there is always a physical limit. In our case this limit is set by diffraction as discussed above: We expect the self-similarity of the mode profile to hold down to transverse distances of the order of $1/N$. In the limit $N \to \infty$ (e.g., $\lambda \to 0$) the magnified imaging becomes perfect in the sense of geometric optics (i.e., no diffractive spreading), leading to an ideal fractal.

To prove that the eigenmode intensity profiles in Fig. 2 are indeed fractals, we now calculate the fractal dimension D of the curves in Fig. 2. The fractal dimension measures the degree of roughness of a self-similar curve. To determine D we use the so-called box-counting method[6]: We cover the curve with a grid of boxes of size $d \times d$ and count, for each box size d, the number of boxes n that are needed to cover the curve fully. Then the fractal dimension follows as

$$D = \lim_{d \downarrow 0} \frac{\log(n)}{\log(1/d)}\cdot \tag{2}$$

If this recipe is applied to a smooth curve one will find $D = 1$, since $n = 1/d$ for small d, but for a fractal

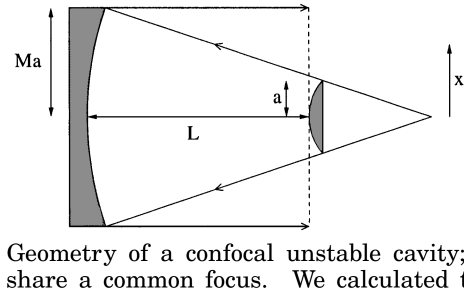

Fig. 1. Geometry of a confocal unstable cavity; the two mirrors share a common focus. We calculated the mode profiles in the aperture plane indicated by the dashed line.

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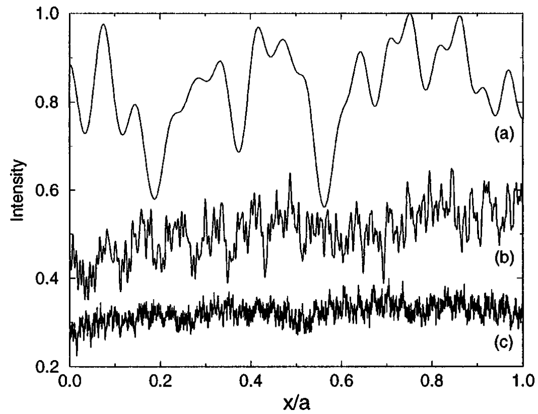

Fig. 2. Intensity profiles for a strip resonator for different Fresnel numbers N. The resonator has a magnification $M = 2$. (a) $N = 10$, (b) $N = 100$, and (c) $N = 1000$. Curve (b) is shifted downward by an amount 0.35 and (c) by 0.6 to avoid overlap with curve (a).

one will find a noninteger value for D, since a fractal has structure on any scale. We apply this method to curve (c) in Fig. 2, the curve that we expect to be fractal over the largest dynamic range. The result is given in Fig. 3, which shows n versus d.

From Fig. 3 we conclude that the mode profile has indeed fractal structure corresponding to a fractal dimension $D = 1.6$. The dynamic range over which the mode profile is fractal extends to $d \approx 2 \times 10^{-3}$, which is indeed approximately $1/N$. This result shows that diffraction is the cause of the finite dynamic range of the fractal, as expected. (Note that we determined the slope by using only the points for which $2 \times 10^{-3} < d < 10^{-1}$, since one expects no fractal behavior outside this range: One needs small d for the box-counting method to converge, but on a scale smaller than $1/N$ the mode profiles are smoothed by diffraction.) We also found that the field amplitude profile has the same fractal dimension as the field intensity profile, as it should, since this is a general mathematical result.[7]

We checked our calculation of D, using an independent method based on analysis of the spatial-frequency power spectrum of the mode profiles. When the power spectrum follows a power law the exponent can be related to the fractal dimension D.[7] This approach yielded results identical to those obtained with the box-counting method.

What determines D? We calculated D for different values of the Fresnel number N (10–1000) and found that D is independent of N (always $D = 1.6$). We found a very weak dependence on M ($D = 1.59-1.51$ for $M = 2-8$), but it seems that this can be attributed to the limited accuracy of the box-counting method owing to the finite size of the data set.[8] Only the dynamic range over which the mode profiles are fractal was found to depend on M and N, as expected. We also checked whether D changes as a result of fine tuning of N in the range 100–101, since it is known that in each unit interval of N the mode profiles can change quite drastically[3,9]; however, no influence on D was found.

Now that we have discussed the strip-resonator case an intriguing question is: What will D be for a cavity with a two-dimensional aperture? The case of a square aperture is trivial, since the eigenmode then is simply the product of two eigenmodes, one for the x direction and one for the y direction, as we found for the one-dimensional strip resonator (Fig. 2). So this case yields $D = 1.6$ for the square aperture. Note that the origin, defined by the optical axis of the cavity, is also the center of magnification; therefore the intensity profile on any line through the origin (i.e., not only lines parallel to the x or the y axis) has a fractal structure with $D = 1.6$, as we have checked. For a circular aperture the diffraction integral is one dimensional, allowing straightforward computation[5]; again we restricted ourselves to the lowest-loss mode. The result is shown in Fig. 4.

Now we find a fractal dimension $D = 1.3$; again this was found to be independent of N and M. The difference between the square and the circular

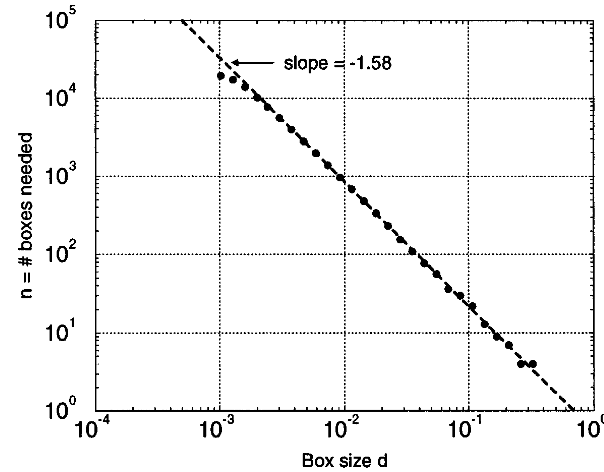

Fig. 3. Application of the box-counting method to a strip resonator, showing the number of boxes of size d needed to cover curve (c) in Fig. 2. The straight line is a fit with a slope of -1.58 ± 0.03.

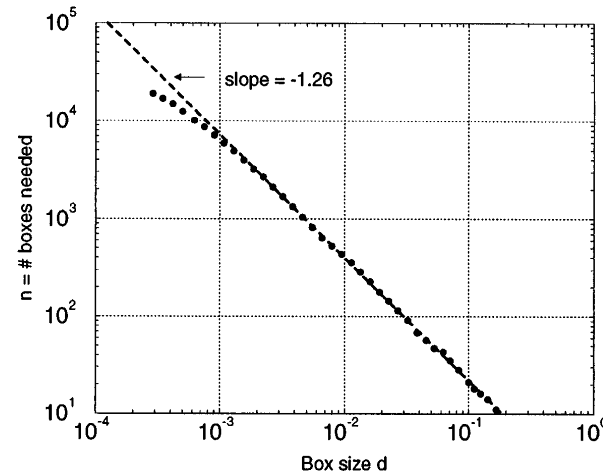

Fig. 4. Application of the box-counting method to an unstable resonator with a circular aperture, $M = 2$, and $N = 1000$, showing the number of boxes of size d needed to cover the intensity mode profile of the lowest-loss mode. The straight line is a fit with a slope of -1.26 ± 0.03.

aperture shows that D depends on the aperture shape. To investigate this shape dependence further one must be able to calculate mode profiles for arbitrarily shaped apertures, which is far from trivial; nevertheless we have initiated calculations on polygon-shaped apertures and hope to report on this in the near future.

In conclusion, we have shown that the eigenmodes of a hard-edged unstable-cavity laser have fractal structure over a dynamic range limited by diffraction, i.e., by the Fresnel number. The origin of the fractal structure can be understood intuitively: The eigenmode is invariant under magnification, which is the definition of a fractal. The fractal dimension D was found to depend on the shape of the aperture: We found $D = 1.6$ for a square aperture and $D = 1.3$ for a circular aperture. It remains an open question why these values are what they are. Note, however, that the ratio of these two values is close to the diffraction spot width difference between a square and a circular aperture.

It is known that unstable-cavity lasers have interesting excess quantum-noise properties[9–11] that are strongly dependent on the aperture shape[12]; therefore it would be interesting to see what the consequence of the fractal nature of the eigenmodes is for excess quantum noise in these lasers.

Finally, we note that fractal properties of fields diffracted by nonfractal objects have also appeared in other systems. Recently Berry and Klein discussed the so-called fractal Talbot effect[8]: When a binary grating is illuminated by a collimated beam of light, the diffracted field contains a fractal structure. Essential here are the hard edges of the grating slits and the periodicity of the grating. A related unexpected appearance of fractals is in the wave-mechanical problem of a particle in a box; because of the square well potential (i.e., the hard edges), both the time and spatial evolution of the probability density exhibit fractal behavior, giving $D = 1.5$.[7] We believe that the mode structure of an unstable-cavity laser is related to the wave dynamics of a particle in a box, which now has a convex bottom (corresponding to the curvature of the cavity mirror).

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