16

BPM application in time-domain problems

The purpose of the following chapter is to place beam propagation techniques in a broader context. We will show how BPM techniques can be applied to simulation of optical pulses.

A classical formulation of time-domain beam propagation methods is briefly discussed first. This is an approach that applies techniques of z-propagated BPM to evolution in time direction. It is applicable to situations in which dispersive properties of the media that constitute the modeled device are not playing a too important role, and when spectral broadening of the initial waveform is small.

A somewhat closer examination follows for methods that are generally applicable to arbitrarily complex spectral dynamics, but that adhere to the spirit of BPM in the sense that optical pulse propagation has a well-defined direction.

Because detailed discussion of time-domain issues is beyond the scope of our course, this chapter is meant to provide a concise overview, emphasizing that many of BPM techniques are directly applicable as components of time-domain simulations.

16.1 Time-domain beam propagation

Beam propagation techniques are most often applied to problems of directed light propagation in waveguides. When interfaces are present that are perpendicular to the direction of propagation, additional computational problems arise, and the fields must be treated as composed of waves that propagate both forward and backward. Besides the so-called bi-directional BPM, time-domain versions of beam propagation have been also developed. Here we summarize their typical properties.

The central idea is that of utilizing time as the propagation variable instead of the coordinate z. Because of the symmetry between t and z, most of the techniques can be directly adopted.

Optical field is represented with a "time-domain" envelope, by factoring out its carrier-wave time dependence,

$$E(y, z, t) = A(y, z, t)e^{-i\omega_0 t} , \tag{16.1}$$

which is of course closely related to the BPM envelope approach. If we assume that the geometry of the waveguide does not change along direction x, we can start from the wave equation written for the slowly varying envelope

$$\frac{\partial^2 A}{c^2 \partial t^2} - 2i\omega_0 \frac{\partial A}{c^2 \partial t} + \frac{\omega_0^2}{c^2} A - \Delta A + \nabla\nabla . A = 0 . \tag{16.2}$$

Next, the first term is usually neglected on grounds of the spectrum of the pulse being narrow in comparison to the carrier wave frequency ω_0. Alternatively, techniques akin to wide-angle BPM can be applied that can include the second time derivative. However, such an approach makes little sense due to other approximations that this kind of method normally needs.

This has to do with the fact that whenever a broad spectrum results from the interactions with media, propagation should be modeled with proper account for the linear dispersive properties of the waveguide materials. Note that the above equation does not show frequency-dependent refractive index, and it is for a reason, and namely because treatment of chromatic dispersion in time domain is not simple.

It is possible to design methods that reflect dispersive properties of material in the numerical wave propagation. After all, many Maxwell solvers rely on some way or another to include frequency dependent material response in their evolution in time. However, these approaches are not without difficulties, they require significant additional memory and computing time — readers should recall our discussion in the introductory sections of this course.

Whether the second time-derivative is included or not, the envelope wave equation can be treated by any of the BPM family of methods, be they based on finite-difference of finite-element discretizations. One particular advantage of this approach is in devices consisting of different longitudinal sections, or ones that contain optical gratings, that result in truly bi-directional wave propagation.

16.2 Directional pulse propagation equations

16.2.1 Nonlinear, structured medium

We consider a non-magnetic, isotropic dispersive medium, with the dielectric permittivity $\epsilon(x, y, \omega)$, which only depends on coordinates x and y and angular frequency ω. We further assume there are no free charges or currents, and that constitutive relations can be written in terms of polarization, which accounts for all medium properties except the linear permittivity $\epsilon(\omega)$

$$\mathbf{P} = \mathbf{P}(x, y, \{\mathbf{E}(x, y, t)\}) \ . \tag{16.3}$$

In this way, we want to separate the medium properties that are usually treated in the BPM regime, i.e. refractive index profile throughout the device, from those that can be "added" as a response of the medium to the presence of the optical field. The latter will most often represent nonlinear medium properties.

The concrete form of $\mathbf{P}$ is unimportant for the present chapter, as we want to elucidate general properties of optical propagation equations which do not depend on details of light-matter interactions. However, as a specific example, the reader can think of the instantaneous optical Kerr effect in which the local refractive index responds to the squared "length" of the electric field intensity vector:

$$\begin{aligned}&\mathbf{P}_{Kerr}(x, y, \{\mathbf{E}(x, y, t)\}) \\ &\quad = 2\epsilon_0 \bar{n}_2(x, y)(E_x^2 + E_y^2 + E_z^2)\mathbf{E}.\end{aligned} \tag{16.4}$$

Here, $\bar{n}_2(x, y)$ stands for the nonlinear index, which in general depends on position. This is the fully vectorial form of the instantaneous Kerr effect which we utilized as a "testing ground" for our beam-propagation algorithms in Section XXX.

It should be noted that current density also plays a role of a source in Maxwell equations, and it should be included in a general treatment. Because it gives rise to terms that are analogous to those we are about to encounter with the polarization response, we leave it to the reader to perform these straightforward generalizations.

16.2.2 Time-domain problem as a coupled system of BPM simulations

In numerical simulations which deal with temporal representations of optical fields, analytic signals of the electric field are often utilized as the "native" representation. For example, the electric field is the real part of its analytic signal:

$$\mathbf{E} = \mathrm{Re}\{\boldsymbol{E}(x,y,z,t)\}, \tag{16.5}$$

which has non-zero spectral power only for positive frequencies:

$$\boldsymbol{E}(x,y,z,t) = \int_0^\infty d\omega \boldsymbol{E}(x,y,z,\omega)e^{-i\omega t}. \tag{16.6}$$

In what follows we will mostly work in this spectral representation, so unless specified otherwise, the optical field will be referred to through the above "spectrum" $\boldsymbol{E}(x,y,z,\omega)$. This is very much in line with the beam-propagation angle of view. Indeed, for each fixed angular frequency, we have a stand-alone BPM problem to solve. Light-medium interactions will in general induce interactions between different spectral slices, so that these seemingly independent BPM simulations must be somehow coupled. It is the main subject of this chapter to show how this can be done without unnecessary approximations, and in ways suitable for practical simulations.

Clearly, if no nonlinear interactions affect the optical field propagation, there is no change to the spectrum of the waveform, and each frequency component can be treated independently. As a result, pulse simulation is trivially reduced to a set of un-coupled beam-propagation problems. It can be therefore safely assumed that all linear light-matter interactions have been included in the frequency-dependent refractive index.

The situation changes whenever new frequency components can rise from "other" (i.e. nonlinear) light-matter interactions. The important aspect that affect how simulators are built is that the response of the medium is best described in time-domain; it is calculated with the history of the electric field serving as an input. The simple example of the instantaneous Kerr effect in the previous section illustrates the point that this calculation is most effectively executed in real-time representation. However, since the native field representation is in the spectral domain, electric field as a function of time must first be evaluated to calculate the medium response at each spatial point. Once the nonlinear polarization has been calculated, it can be transformed back into spectral representation,

$$\mathbf{P} = \mathbf{P}(x,y,\{\mathbf{E}(x,y,t)\}) \to \mathbf{P}(x,y,\omega) \ ,$$

and this is the quantity that will enter in our discussion that follows. We will keep in mind that it depends implicitly on the electric field history at (x,y), but for the sake of brevity, this dependence will be omitted from our notations.

In short, the BPM sub-problems that constitute a time-dependent optical pulse simulations are only coupled through the nonlinear light-matter interactions. How the latter are modeled is utterly irrelevant to the rest of the propagation problem — in fact they should be implemented as a black box, canned routines so that the main simulator engine can be built independently of concrete nonlinear processes are to be included.

With the nonlinear interaction "delegated" to a suitable plug-in routines, the crucial ingredient is to set up propagation equations that can "separate" linear and nonlinear properties. It is of desirable to achieve such separation with as little physical assumptions as possible to ensure as wide applicability as possible. This problem is discussed in the following sections.

16.2.3 Directional Pulse Propagation Equations

Each of the spectral slices that constitute the full optical waveform obeys the wave equation, accompanied by the divergence constraint:

$$\nabla\nabla\cdot\boldsymbol{E}-\nabla^2\boldsymbol{E}=\frac{\omega^2}{c^2}\left(\epsilon\boldsymbol{E}+\frac{1}{\epsilon_0}\boldsymbol{P}\right)\ ,\ \nabla\cdot\boldsymbol{D}=0 \tag{16.7}$$

The latter can be expressed through the electric field and medium polarization response

$$\nabla\cdot\boldsymbol{D}=\epsilon_0\epsilon\nabla\cdot\boldsymbol{E}+\epsilon_0\boldsymbol{E}\cdot\nabla\epsilon+\nabla\cdot\boldsymbol{P}=0, \tag{16.8}$$

and, consequently, $\nabla\cdot\boldsymbol{E}$ can be written as

$$-\nabla\cdot\boldsymbol{E}=\frac{1}{\epsilon}\boldsymbol{E}_\perp\cdot\nabla_\perp\epsilon+\frac{1}{\epsilon_0\epsilon}\nabla\cdot\boldsymbol{P} \tag{16.9}$$

Inserting this in the wave equation, the transverse (x,y) part of it is rewritten to separate its linear and nonlinear terms

$$-\partial_{zz}\boldsymbol{E}_\perp=\hat{L}\boldsymbol{E}_\perp+\hat{N}_\perp[\boldsymbol{E}]. \tag{16.10}$$

Here, the linear operator $\hat{L}$ derives from the Helmholtz equation. It acts on the transverse electric field vector as follows

$$\hat{L}\boldsymbol{E}_\perp\equiv\frac{\omega^2}{c^2}\epsilon(r_\perp,\omega)\boldsymbol{E}_\perp+\Delta_\perp\boldsymbol{E}_\perp+\nabla\frac{1}{\epsilon}\boldsymbol{E}_\perp.\nabla_\perp\epsilon. \tag{16.11}$$

The nonlinear operator $\hat{N}$ also acts on $\boldsymbol{E}_\perp$,

$$\hat{N}[\boldsymbol{E}]\equiv\frac{\omega^2}{\epsilon_0c^2}\boldsymbol{P}(\boldsymbol{E})+\nabla\frac{1}{\epsilon_0\epsilon}\nabla\cdot\boldsymbol{P}(\boldsymbol{E})\ . \tag{16.12}$$

but in general, also depends on the E_z component, which will be addressed later.

Up to this point, there is no departure from our previous treatment of BPM equations. The reader will surely recall that it was the second derivative with respect to propagation distance z where paraxial and wide-angle methods diverged, and that in neither case the treatment was completely satisfactory from the mathematical point of view. This time, we address the issue more rigorously.

To obtain propagation equations for $\boldsymbol{E}_\perp(z,x,y,\omega)$, we first introduce auxiliary field amplitudes, doubling the number of variables used to describe the electric field:

$$E_i(z,x,y,\omega)=E_i^+(z,x,y,\omega)+E_i^-(z,x,y,\omega), \tag{16.13}$$

where each contribution has its own carrier wave

$$\begin{aligned}E_i^+&=A_i^+(z,x,y,\omega)e^{+i\zeta z}\\E_i^-&=A_i^-(z,x,y,\omega)e^{-i\zeta z},\end{aligned} \tag{16.14}$$

where $i=x,y$ and ζ stands for a parameter to be chosen freely. It resembles the reference wavenumber that appears throughout BPM treatment and which is usually meant to eliminate the fastest changes in simulated fields. In this sense, $E_i^\pm$ may be viewed as "slow" amplitudes, perhaps representing waves propagating forward and backward, but it must be kept in mind that we have said nothing so far that would guarantee that these auxiliaries would in fact evolve slowly with z. Because no ζ appears in Maxwell equations, we call ζ a reference wavenumber to emphasize the fact that it has no physical meaning by itself. We will see that once it serves its auxiliary role, it will disappear from the final results.

So, in general the positive and negative wavenumber parts $E_i^{\pm}$ of the field are not the forward and backward propagating waves. In fact, both $E_i^{\pm}$ can contribute to waves propagating in the positive and negative z directions. This is because we have not restricted how fast $A_i^{\pm}$ can change with z, and they could evolve so fast that their variation would completely override the exponential factors $e^{\pm i\zeta z}$ accompanying them.

As mentioned above, at this point we have twice as many degrees of freedom to represent our optical field as is physically required. This we will take advantage of by requiring that $E_i^{\pm}$ satisfy a relation of our choice, which will correct the number of independent variables. Inspired by what is done in method of undetermined coefficients for ordinary differential equations, we impose an additional constraint in the form

$$e^{+i\zeta z}\partial_z A_i^{+}(z,x,y,\omega) + e^{-i\zeta z}\partial_z A_i^{-}(z,x,y,\omega) = 0 \tag{16.15}$$

As in the variation of constants method, this representation eliminates the second derivatives when one evaluates $\partial_{zz}E$. Because of the constraint, the first derivative simplifies to

$$\partial_z E_i = i\zeta(E_i^{+} - E_i^{-}) \tag{16.16}$$

and the second derivative is

$$\partial_{zz}E_i = -\zeta^2 E_i + i\zeta e^{+i\zeta z}\partial_z A_i^{+} - i\zeta e^{-i\zeta z}\partial_z A_i^{-} \ . \tag{16.17}$$

Thus, the second ∂_{zz} derivative can be obtained as a combination of two auxiliaries and their first -order derivatives. Using the following in the wave equation (16.10) together with the constraint of Eq. (16.15), it is straightforward to show that the evolution equations for the auxiliary amplitudes $A^{\pm}$ read as

$$\partial_z A_i^{\pm} = \frac{\pm i}{2\zeta} e^{\mp i\zeta z}\left[(\hat{L}-\zeta^2)\boldsymbol{E}_{\perp} + \hat{N}[\boldsymbol{E}]\right] \ , \tag{16.18}$$

and in language of auxiliary E-fields they are

$$\partial_z E_i^{+} = +i\zeta E_i^{+} + \frac{i}{2\zeta}\left[(\hat{L}-\zeta^2)\boldsymbol{E}_{\perp} + \hat{N}[\boldsymbol{E}]\right] \tag{16.19}$$

$$\partial_z E_i^{-} = -i\zeta E_i^{-} - \frac{i}{2\zeta}\left[(\hat{L}-\zeta^2)\boldsymbol{E}_{\perp} + \hat{N}[\boldsymbol{E}]\right] \ . \tag{16.20}$$

The next step is to identify the true forward and backward propagating fields. In a matrix notation, the propagation equations read

$$\partial_z \begin{pmatrix} \boldsymbol{E}_{\perp}^{+} \\ \boldsymbol{E}_{\perp}^{-} \end{pmatrix} = i \begin{pmatrix} \zeta + \frac{\hat{L}-\zeta^2}{2\zeta} & +\frac{\hat{L}-\zeta^2}{2\zeta} \\ -\frac{\hat{L}-\zeta^2}{2\zeta} & -\zeta - \frac{\hat{L}-\zeta^2}{2\zeta} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_{\perp}^{+} \\ \boldsymbol{E}_{\perp}^{-} \end{pmatrix} + \frac{i}{2\zeta}\begin{pmatrix} +\hat{N}_{\perp}[\boldsymbol{E}] \\ -\hat{N}_{\perp}[\boldsymbol{E}] \end{pmatrix} . \tag{16.21}$$

Having separated the linear and nonlinear part of the evolution operator, the forward-backward division will be defined with respect to the linear system. Two projector operators can be constructed from the Helmholtz operator $\hat{L}$ and its square root $\hat{L}^{\frac{1}{2}}$:

$$\mathcal{P}_F \equiv \frac{\hat{L}^{-\frac{1}{2}}}{4\zeta}\begin{pmatrix} +(\zeta+\hat{L}^{\frac{1}{2}})^2 & +(\hat{L}-\zeta^2) \\ -(\hat{L}-\zeta^2) & -(\zeta-\hat{L}^{\frac{1}{2}})^2 \end{pmatrix} \tag{16.22}$$

$$\mathcal{P}_B \equiv \frac{\hat{L}^{-\frac{1}{2}}}{4\zeta}\begin{pmatrix} -(\zeta-\hat{L}^{\frac{1}{2}})^2 & -(\hat{L}-\zeta^2) \\ +(\hat{L}-\zeta^2) & +(\zeta+\hat{L}^{\frac{1}{2}})^2 \end{pmatrix} \tag{16.23}$$

It is a lengthy but straightforward calculation to show that these operators have properties expected of projectors. First, they are idempotent

$$\mathcal{P}_F^2 = \mathcal{P}_F \quad \mathcal{P}_B^2 = \mathcal{P}_B \ , \tag{16.24}$$

and, second, they constitute a unity decomposition such that

$$\mathcal{P}_F + \mathcal{P}_B = 1 \ , \ \mathcal{P}_F \mathcal{P}_B = \mathcal{P}_F \mathcal{P}_B = 0. \tag{16.25}$$

Importantly, these projectors also commute with the linear evolution operator in Eq. (16.21). Thus, they can be used to project out the true forward and backward propagating field components. If the total field is given in terms of the auxiliary amplitudes $\boldsymbol{E}_\perp^\pm$, then the forward portion of the wave is obtained as

$$\begin{aligned} E_F &= \begin{pmatrix} 1 \ 1 \end{pmatrix} \mathcal{P}_F \begin{pmatrix} \boldsymbol{E}_\perp^+ \\ \boldsymbol{E}_\perp^- \end{pmatrix} \\ &= \frac{1}{2} \left[(\boldsymbol{E}_\perp^+ + \boldsymbol{E}_\perp^-) + \hat{L}^{-\frac{1}{2}} \zeta (\boldsymbol{E}_\perp^+ - \boldsymbol{E}_\perp^-) \right] . \end{aligned} \tag{16.26}$$

This expression contains the reference wavenumber ζ, which might suggest that it depends on our artificial split of the field in Eq. (16.13). However, because of Eq. (16.16), the above expression reduces to

$$E_F = \frac{1}{2} \boldsymbol{E}_\perp - \frac{i}{2} \hat{L}^{-\frac{1}{2}} \partial_z \boldsymbol{E}_\perp \tag{16.27}$$

and the backward propagating component is obtained as

$$E_B = \frac{1}{2} \boldsymbol{E}_\perp + \frac{i}{2} \hat{L}^{-\frac{1}{2}} \partial_z \boldsymbol{E}_\perp \ , \tag{16.28}$$

which is free of the artificial quantity ζ. These equations deserve a closer look. They provide a general recipe to split a given vector field into forward and backward propagating parts. In BPM, we normally assume that the initial condition coincides with E_F, and now we can see that such an assumption implies that the derivative with respect to the propagation distance is also implicitly given.

Our goal are pulse evolution equations in terms of the directional fields E_F, E_B. Because the projector operators are z-independent, they commute with ∂_z. The simplest way forward is therefore to apply them directly to the propagation equations, and then sum up auxiliary parts to obtain physical field. The left hand side is

$$\partial_z E_\perp^F = \begin{pmatrix} 1 \ 1 \end{pmatrix} \mathcal{P}_F \begin{pmatrix} \partial_z \boldsymbol{E}_\perp^+ \\ \partial_z \boldsymbol{E}_\perp^- \end{pmatrix} \tag{16.29}$$

$$\partial_z E_\perp^B = \begin{pmatrix} 1 \ 1 \end{pmatrix} \mathcal{P}_B \begin{pmatrix} \partial_z \boldsymbol{E}_\perp^+ \\ \partial_z \boldsymbol{E}_\perp^- \end{pmatrix} . \tag{16.30}$$

Inserting the right hand side from Eq. (16.21) and using the projector properties of Eqs. (16.24) and (16.25), we obtain a pair of coupled equations for the forward and backward fields

$$\begin{aligned} \partial_z E_\perp^F &= +i\sqrt{\hat{L}} E_\perp^F + \frac{i}{2\sqrt{\hat{L}}} \hat{N}_\perp [E^F + E^B] \\ \partial_z E_\perp^B &= -i\sqrt{\hat{L}} E_\perp^B - \frac{i}{2\sqrt{\hat{L}}} \hat{N}_\perp [E^F + E^B] \end{aligned} \tag{16.31}$$

This is a coupled pair of pulse propagation equations. It is exact, but not suitable for numerical simulations in this very form, because the forward and backward fields are fast quantities that change on the length-scale of a single wavelength. This is why it is better to transform this system with the help of integrating factor the same way we have used it in chapter 6. Indeed, we can select amplitudes which will exhibit evolution only if some nonlinearity is present:

$$E_\perp^F = e^{+i\sqrt{\hat{L}}z} A_\perp^F(z)\ ,\ E_\perp^B = e^{-i\sqrt{\hat{L}}z} A_\perp^B(z)\ . \tag{16.32}$$

Note that these spectral amplitudes are different from the auxiliaries introduced in previous sections. In this representation, Eq. (16.31) reads

$$\partial_z A_\perp^F = \frac{+i}{2\sqrt{\hat{L}}} e^{-i\sqrt{\hat{L}}z} \hat{N}_\perp [e^{+i\sqrt{\hat{L}}z} A^F + e^{-i\sqrt{\hat{L}}z} A^B]$$
$$\partial_z A_\perp^B = \frac{-i}{2\sqrt{\hat{L}}} e^{+i\sqrt{\hat{L}}z} \hat{N}_\perp [e^{+i\sqrt{\hat{L}}z} A^F + e^{-i\sqrt{\hat{L}}z} A^B]$$

We have encountered this is in a similar form in chapter 6 where we have deduced that if the above equations are satisfied, then the field also obeys the wave equations. Together with the material shown in this section, we have established that these equations are in fact *equivalent* to the wave equation.

Our treatment shows that the forward and backward propagating waves are mutually coupled through the nonlinearity of the medium. In order to obtain a time-resolved counterpart of beam propagation (i.e. one that works solely with directional waves), we must adopt an approximation which will allow us to reduce the full system to a single, unidirectional equation.

16.2.4 Unidirectional approximation

In the spirit of BPM, we wish to eliminate all waves that propagate in the “wrong” direction. Derivations in this chapter showed clearly that this is not possible in general. Interactions with the medium will always induce mutual coupling, and even when a waveform is carefully prepared as uni-directional, it will generate light back-scattered in the opposite direction. On the other hand, experience tells us that in many situations this is of no practical concern because such light is very weak.

We therefore assume that the nature and strength of nonlinearity is such that only negligible backward propagating fields are generated. Then, the nonlinear term can be approximated as

$$\hat{N}_\perp [e^{+i\sqrt{\hat{L}}z} A^F + e^{-i\sqrt{\hat{L}}z} A^B] \approx \hat{N}_\perp [e^{+i\sqrt{\hat{L}}z} A^F] \tag{16.33}$$

and the system can be restricted to only the forward-propagating field:

$$\partial_z A_\perp^F(r_\perp, \omega, z) = +\frac{i}{2\sqrt{\hat{L}}} e^{-i\sqrt{\hat{L}}z} \hat{N}_\perp [e^{+i\sqrt{\hat{L}}z} A^F]. \tag{16.34}$$

This is a uni-directional pulse propagation equation that encompasses a wide variety of practically useful cases. In the context of this course, it should be appreciated that the beam propagation techniques are needed for practical solution of these equations.

The numerical strategy to solve this kind of coupled system was described in the section dealing with spectral representations and an integrating factor approach. It builds on the ODE-based method for solving UPPE systems and combines it with a wide-angle beam-propagation solver used to evaluate the linear propagator $\exp(i\hat{L}^{1/2}z)$.

16.2.5 Special case: pulse propagation in a homogeneous medium

To illustrate how the general framework applies to concrete situations, we consider three special cases. First, let us look at pulse propagation in a bulk medium, gaseous, or condensed. This is in fact the simplest case and a straightforward extension of the method discussed in chapter 6.

Plane waves are eigenfunctions of the Helmholtz operator $\hat{L}$ and the corresponding propagator $\exp(i\hat{L}^{1/2}z)$ can be handled in the (spatial) spectral representation — this is nothing but application FFT-BPM or DHT-BPM methods. In the plane-wave basis the linear propagator reduces to multiplication by a phase factor given by the propagation constant $k_z(\omega, k_\perp)$:

$$e^{-i\sqrt{\hat{L}}z} = e^{-ik_z(\omega,k_\perp)z}.$$

It is therefore sufficient to Fourier-transform Eq. (16.34) from the (x, y) space to the transverse wavenumber space (k_x, k_y) to obtain

$$\partial_z A^F_\perp(k_\perp, \omega, z) = +\frac{i}{2k_z} e^{-ik_z z} \hat{N}_\perp[e^{+ik_z z} A^F], \tag{16.35}$$

where one has to keep in mind that k_z is the propagation constant of a plane wave that depends on its frequency, transverse wavenumber and of course on the medium in which the wave propagates. This a system of ODEs that is solved in practical simulations.

Because the vector properties of the electric field are rather implicit in the representation based on spectral amplitudes $A^F_\perp(k_\perp, \omega, z)$, it is instructive to write the corresponding equations for the physical electric field. Using Eq. (16.32)

$$\partial_z E^F_\perp(k_\perp, \omega, z) = +ik_z E^F_\perp(k_\perp, \omega, z) + \frac{i}{2k_z} \hat{N}_\perp[E^F], \tag{16.36}$$

and expressing $\hat{N}$ in terms of polarization we obtain

$$\partial_z E^F = ik_z E^F + \frac{i}{2k_z}\left[\frac{\omega^2}{\epsilon_0 c^2}\boldsymbol{P}(\boldsymbol{E}) - \frac{1}{\epsilon_0 \epsilon}\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{E})\right]. \tag{16.37}$$

Only the transverse components in this equation constitute the evolution system, but in this full-vector form, it is easy to see that the operator acting on the polarization term produces the transverse part of the nonlinear response, namely

$$\left[\frac{\omega^2}{\epsilon_0 c^2}\boldsymbol{P}(\boldsymbol{E}) - \frac{1}{\epsilon_0 \epsilon}\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{E})\right] = \frac{\omega^2}{\epsilon_0 c^2}\left[1 - \frac{\boldsymbol{k}\boldsymbol{k}\cdot}{k^2}\right]\boldsymbol{P}(\boldsymbol{E}). \tag{16.38}$$

The projector operator in the square brackets can be replaced by a sum over projectors on the polarization vectors $\boldsymbol{e}_s$

$$\left[1 - \frac{\boldsymbol{k}\boldsymbol{k}\cdot}{k^2}\right] = \sum_s \boldsymbol{e}_s\boldsymbol{e}_s\cdot \tag{16.39}$$

Naturally, any choice of polarization vectors can be used here, in particular ones representing circularly or linearly polarized waves. This projection property follows solely form the fact that both polarization vectors are perpendicular to each other and also to the wave vector $\boldsymbol{k}$.

Using the above projector in Eq. (16.38) and inserting it into Eq. (16.37), we obtain

$$\partial_z E^F_\perp(z, \omega, k_\perp) = ik_z E^F_\perp + \frac{i\omega^2}{2\epsilon_0 c^2 k_z}\sum_s \boldsymbol{e}^\perp_s \boldsymbol{e}_s \cdot \boldsymbol{P}(\boldsymbol{E}), \tag{16.40}$$

which is a fully vectorial propagation equation for the transverse part of the forward-propagating vector field E^F. The sole approximation it assumes is that nonlinear interactions do not produce strong backward propagating radiation.

16.2.6 Special case: pulse propagation in optical fibers

The traditional way to derive propagation equations for fibers are based on modal projections, which in turn utilize orthogonal properties of electromagnetic modes. However, most of the time one projects the field within the waveguide onto a single, fundamental guided mode — this is the case we will restrict or attention to. With only a single mode to deal with, the derivation can follow the same reasoning as the effective index method. In the effective index method, we assumed that the shape of the electromagnetic field in one of the directions perpendicular to the propagation is fixed and given by a guided mode of the structure. This can be extended to fiber by restricting the field profile in both transverse dimensions to the fundamental mode of the fiber.

We could start the deviation from the spatial-spectral decomposition and Helmholtz equations for all frequency components (see sect. XXX). But a more convenient point of departure is the general uni-directional equation we have just derived:

$$\partial_z A_\perp^F(r_\perp,\omega,z) = +\frac{i}{2\sqrt{\hat{L}}} e^{-i\sqrt{\hat{L}}z} \hat{N}_\perp[e^{+i\sqrt{\hat{L}}z} A^F] \; . \tag{16.41}$$

Here we use our assumption that at all frequencies ω and for all propagation distances z, the transverse shape of the field is that of the fundamental mode of the fiber $\mathcal{M}(r_\perp)$:

$$A_\perp^F(r_\perp,\omega,z) \equiv \mathcal{M}(r_\perp) A(\omega,z) \; . \tag{16.42}$$

$$\mathcal{M}(r_\perp)\partial_z A(\omega,z) = +\frac{i}{2\sqrt{\hat{L}}} e^{-i\sqrt{\hat{L}}z} \hat{N}_\perp[e^{+i\sqrt{\hat{L}}z} A^F] \tag{16.43}$$

According to our assumption, every time the linear propagator $e^{-i\sqrt{\hat{L}}z}$ acts on a propagating waveform, it only "sees" the fundamental mode and that is why its action reduces to multiplication by the eigenvalue corresponding to the fundamental mode, which is nothing but the propagation constant $\beta(\omega)$ of the latter. So, we can write the above as

$$\mathcal{M}(r_\perp)\partial_z A(\omega,z) = +\frac{i}{2\beta(\omega)} e^{-i\beta(\omega)z} \hat{N}_\perp[\mathcal{M}(r_\perp) e^{+i\beta(\omega)z} A(\omega,z)] \tag{16.44}$$

This equation still carries the spatial profile of the field, which is not useful because it gives no new information. To get rid of it, we multiply the whole equation by the modal field (which is always possible to choose as real-value, and that is why non complex conjugation is needed), and integrate over the whole transverse domain x, y:

$$\left(\int d^2 r_\perp \mathcal{M}^2(r_\perp)\right) \partial_z A(\omega,z) = +\frac{i}{2\beta(\omega)} e^{-i\beta(\omega)z} \int d^2 r_\perp \mathcal{M}(r_\perp) \hat{N}_\perp[\mathcal{M}(r_\perp) e^{+i\beta(\omega)z} A(\omega,z)] \tag{16.45}$$

To proceed, one must know something about the nature of nonlinearity. It will be assumed that the nonlinearity order is three, as is most common for optical fibers. This naturally includes both Kerr and stimulated Raman effects. It should be obvious what changes when the nonlinear interactions exhibit different orders. Recall that the nonlinear response is a Fourier transform (from time to angular frequency),

$$\hat{N}[\boldsymbol{E}] \equiv FT\left\{\frac{\omega^2}{\epsilon_0 c^2}\boldsymbol{P}(\boldsymbol{E}) + \nabla\frac{1}{\epsilon_0\epsilon}\nabla\cdot\boldsymbol{P}(\boldsymbol{E})\right\} \; . \tag{16.46}$$

This can be further simplified — we have in fact already negleted the second term, because of the effective index approximation. So far we have not specified the nature of the nonlinearity, appart from the fact that it is of the third order. For concreteness, let it be instantaneous Kerr effect, for which

$$\hat{N}[\boldsymbol{E}] \equiv FT\{\frac{\omega^2}{\epsilon_0 c^2}\boldsymbol{P}(\boldsymbol{E})\} = 2n(\omega)n_2\frac{\omega^2}{c^2}\mathcal{M}^3(r_\perp)FT\{E^3(t)\} \ . \tag{16.47}$$

For this cubic nonlinearity we get

$$\left(\int d^2r_\perp \mathcal{M}^2(r_\perp)\right)\partial_z A(\omega,z) = +\frac{in_2 n(\omega)\omega^2 e^{-i\beta(\omega)z}}{c^2\beta(\omega)}\left(\int d^2r_\perp \mathcal{M}^4(r_\perp)\right)FT[(FT^{-1}[e^{+i\beta(\omega)z}A(\omega,z)])^3] \tag{16.48}$$

Next we choose an appropriate unit to represent the optical field. Power, rather than intensity, is most often used in the fiber context. So we write the spectral amplitude as

$$A(\omega,z) = \frac{P_0\mathcal{A}(\omega,z)}{\left(\int d^2r_\perp \mathcal{M}^2(r_\perp)\right)^{1/2}}$$

such that $|FT[\mathcal{A}(\omega,z)](t)|^2$ is the instantaneous power (in P_0 W) of the optical pulse time slice t. In these units, the propagation equation can be rewritten as follows

$$\partial_z\mathcal{A}(\omega,z) = +\frac{in_2P_0 n(\omega)\omega^2 e^{-i\beta(\omega)z}}{c^2\beta(\omega)}\frac{\left(\int d^2r_\perp \mathcal{M}^4(r_\perp)\right)}{\left(\int d^2r_\perp \mathcal{M}^2(r_\perp)\right)^2}FT[(FT^{-1}[e^{+i\beta(\omega)z}\mathcal{A}(\omega,z)])^3] \tag{16.49}$$

The ratio of integrals over transverse space has the meaning of an effective area, an that is the only quantity that, at this level, characterizes the spatial modal property of the fiber. One can also approximate

$$\frac{n(\omega)}{\beta(\omega)} = \frac{n(\omega)c}{\omega n_{eff}(\omega)} \approx \frac{c}{\omega} \ .$$

This is not a big compromise, at least not at this stage — we have assumed that modal field $\mathcal{M}(r_\perp)$ is frequency independent, and that means that details of how the nonlinear coupling depends on wvwelength are partially lost. As a result the pulse propagation equation is

$$\partial_z\mathcal{A}(\omega,z) = +\frac{i\omega}{c}\frac{n_2P_0}{A_{eff}}e^{-i\beta(\omega)z}FT[(FT^{-1}[e^{+i\beta(\omega)z}\mathcal{A}(\omega,z)])^3] \ . \tag{16.50}$$

Most often, one wants to use a moving frame of reference, such that the optical pulsed waveform remains approximately centered somewhere around the center of the temporal computational domain. This is achieved very simply by replacing every $\beta(\omega)$ by

$$\beta(\omega) \to \beta(\omega) - \frac{\omega}{v_{\rm ref}} \ , \tag{16.51}$$

where $v_{\rm ref}$ stands for the moving frame velocity. It is ususally best to choose $v_{\rm ref}$ equal to the group velocity of the initial pulse. If the spectrum of the latter is centered around $\omega = \omega_0$, then

$$\beta(\omega) \to \beta(\omega) - \omega\frac{d\beta}{d\omega_0} \ . \tag{16.52}$$

Equation (16.50) "encapsulates" several other models that are frequently found in the literature. In particular, the so-called generalized nonlinear Schrödinger equation can be derived from here by adopting further approximation. One advantage of this approach is that $\beta(\omega)$ can represent arbitrary fiber and its chromatic dispersion and absorption properties. In practice, this quantity is "imported" into a simulator in a form of tabulated and interpolated function. Second, all the evolution that mujst be resolved in numerics is due to nonlinearity — the linear part of the problem is solved exactly. This is therefore also the form most suitable for numerical solution, as was explained in chapter XXX, section XXX.

16.2.7 Special case: pulse propagation in general waveguides

Throughout this course we have discussed several methods that can deal with the central object that underlines these evolution equations, namely the operator L, its square root, and operator exponential. Obviously, the WA-BPM techniques discussed in chapter XXX apply directly here.

Because the linear propagator is diagonal in angular frequency, its action is equivalent to a set of uncoupled beam-propagation operators. In other words, the action of $\exp(i\hat{L}^{1/2}z)\psi$ only requires one independent BPM-like update for each ω resolved in the simulation. This portion of the algorithm is therefore "embarrassingly parallel," with perfect load balance and no inter-dependencies between calculations performed for different angular frequencies. There are many wide-angle BPM methods available, and at least in principle any of them can be utilized.

For instance, one can evaluate the linear propagator by a Padé approximant. Defining $\beta^2(\omega) \equiv \omega^2\epsilon(\omega)/c^2$, the dominant part of the Helmholtz operator, one writes

$$e^{i\sqrt{\hat{L}}\Delta z} = e^{i\beta\sqrt{1+\hat{X}}\Delta z} = \prod_k \frac{\hat{X}+a_k}{\hat{X}+b_k}. \tag{16.53}$$

The coefficients a_k, b_k depend on Δz and are chosen as to reproduce the Taylor expansion of the left hand side. For example,

$$\frac{4i + (i - \beta\Delta z)\hat{X}}{4i + (i + \beta\Delta z)\hat{X}} e^{i\beta\Delta z} \tag{16.54}$$

is second-order accurate in $\hat{X}$ with an error scaling as $\sim \beta\Delta z\hat{X}^3$. Various higher order approximations can be constructed in the same spirit.

Similar techniques can be used to compute the inverse square root of L that acts on the nonlinear response term in Eq. (16.34). However, this operator can be often approximated by $L^{1/2} \approx \omega n(\omega)/c$.

In conclusion, having implemented the linear propagator as a "BPM-based plug-in," the time-dependent problem of ultra-short pulse propagation through a nonlinear waveguiding structure can be solved as a weakly coupled (large) set of beam-propagation problems.