OPTI-310. Review of useful formulas I.

DISCLAIMER: this document may contain mistakes! It is the user's responsibility to identify them.

Vector identities:

$$\vec{A} \times \vec{A} = 0$$

$$\vec{A}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{A} \times \vec{B}) = 0$$

$$\vec{A}.(\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}).\vec{C}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$$

Operators:
gradient:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

curl:

$$(\nabla \times \vec{E})_x = \partial_y E_z - \partial_z E_y \quad , \quad (\nabla \times \vec{E})_y = \partial_z E_x - \partial_x E_z \quad . \quad (\nabla \times \vec{E})_z = \partial_x E_y - \partial_y E_x$$

Laplacian:

$$\Delta = \nabla.\nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Operator identities:

$$\nabla \times (\nabla T) = 0 \quad , \quad \nabla.(\nabla \times \vec{B}) = 0$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla\nabla.\vec{E} - \Delta\vec{E}$$

$\nabla\psi \rightarrow$ vector , $\nabla.\vec{B} \rightarrow$ scalar , $\nabla\times\vec{E} \rightarrow$ vector , $\Delta\vec{E} \rightarrow$ vector , $\Delta\psi \rightarrow$ scalar

Gauss and Stokes theorems:

$$\int_V \nabla.\vec{F} dV = \int_A \vec{F}.d\vec{S} \quad , \quad \oint_C \vec{F}.d\vec{\ell} = \int_A (\nabla \times \vec{F}).d\vec{S}$$

1D Wave equation:

$$(\partial_{xx} - \frac{1}{v^2}\partial_{tt})\psi(x,t) = 0$$

general solution:

$$\psi(x,t) = f(x - vt) + g(x + vt)$$

3D Wave equation:

$$(\Delta - \frac{1}{v^2}\partial_{tt})\psi(x,y,z,t) = 0$$

Harmonic wave solutions:

$$\psi(\vec{r},t) = A\cos(\vec{k}.\vec{r} - \omega t + \epsilon)$$

Complex representation:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\{\vec{A}\exp[i\vec{k}.\vec{r} - i\omega t + i\epsilon]\}$$

Dispersion relation, period, frequency, wavelength, ...

$$k = \frac{\omega}{v} \quad , \quad T = \frac{1}{\nu} = \frac{2\pi}{\omega} = \frac{2\pi}{kv} = \frac{\lambda}{v}$$

Differential operators acting on harmonic plane waves in complex representation:

$$\nabla\psi = i\vec{k}\psi \quad , \quad \nabla \equiv i\vec{k}$$

$$\nabla^2\psi = -k^2\psi \quad , \quad \nabla^2 \equiv -k^2$$

$$\partial_t\psi = -i\omega\psi \quad , \quad \partial_t \equiv -i\omega$$

$$\partial_{tt}\psi = -\omega^2\psi \quad , \quad \partial_{tt} \equiv -\omega^2$$

Spherically symmetric harmonic waves

$$\psi(r,\theta,\phi,t) = \frac{A}{r}\cos[kr \pm \omega t + \varepsilon]$$

Laplacian of a spherically symmetric function:

$$\Delta\psi(r) = \partial_{rr}\psi(r) + \frac{2}{r}\partial_r\psi(r)$$

Electromagnetism

Lorentz force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Maxwell's equations in integral form (in a dielectric medium with no free charges and currents):

Gauss:

$$\oint_A \vec{E}.d\vec{S} = \oint_A \vec{B}.d\vec{S} = 0$$

Faraday:

$$\oint_C \vec{E}.d\vec{\ell} = -\iint_A \partial_t \vec{B}.d\vec{S}$$

Ampere:

$$\oint_C \vec{B}.d\vec{\ell} = \mu_0 \epsilon \iint_A \partial_t \vec{E}.d\vec{S}$$

Maxwell's equations in differential form (in a dielectric medium with no free charges and currents):

Gauss:

$$\nabla.\vec{E} = \nabla.\vec{B} = 0$$

Faraday:

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

Ampere:

$$\nabla \times \vec{B} = \mu_0 \epsilon \partial_t \vec{E}$$

permitivity, permeability, speed of light:

$$\epsilon = \epsilon_0 \epsilon_r \quad , \quad \epsilon_0 = 8.854 \times 10^{-12} F/m \quad , \quad \mu_0 = 4\pi \times 10^{-7} H/m$$

$$[\epsilon_0] = \frac{F}{m} = \frac{As}{Vm} = \frac{A^2 s^2}{Nm^2} \quad , \quad [\mu_0] = \frac{H}{m} = \frac{Vs}{Am} = \frac{V^2 s^2}{Nm^2}$$

$$\epsilon_0 \mu_0 = 1/c^2$$

Maxwell's equations in differential form (in a dielectric medium with no free charges and currents) in explicit component form:

Gauss:

$$\begin{aligned} \partial_x E_x + \partial_y E_y + \partial_z E_z &= 0 \\ \partial_x B_x + \partial_y B_y + \partial_z B_z &= 0 \end{aligned}$$

Faraday:

$$\begin{aligned} \partial_y E_z - \partial_z E_y &= -\partial_t B_x \\ \partial_z E_x - \partial_x E_z &= -\partial_t B_y \end{aligned}$$

$$\partial_x E_y - \partial_y E_x = -\partial_t B_z$$

Ampere:

$$\begin{aligned} \partial_y B_z - \partial_z B_y &= \mu_0 \epsilon \partial_t E_x \\ \partial_z B_x - \partial_x B_z &= \mu_0 \epsilon \partial_t E_y \\ \partial_x B_y - \partial_y B_x &= \mu_0 \epsilon \partial_t E_z \end{aligned}$$

Maxwell's equations in differential form (in a dielectric, non-magnetic medium):

$$\nabla.\vec{D} = \rho \quad , \quad \nabla.\vec{B} = 0 \quad , \quad \nabla \times \vec{E} = -\partial_t \vec{B} \quad , \quad \nabla \times \vec{H} = \partial_t \vec{D} + \vec{J}$$

Constitutive (material) relations:

$$\vec{B} = \mu_0 \vec{H} \quad , \quad \vec{D} = \epsilon_0 \epsilon_r \vec{E} \equiv \epsilon_0 \vec{E} + \vec{P}$$

Charge conservation:

$$\nabla.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Electromagnetic wave equations

Derivation - wave equation for $\vec{E}$ (taking curl of Faraday):

$$\nabla \times \nabla \times \vec{E} = -\partial_t \nabla \times \vec{B}$$

$$\nabla \nabla.\vec{E} - \Delta \vec{E} = -\partial_t (\mu_0 \epsilon \partial_t E)$$

$$-\Delta \vec{E} = -\mu_0 \epsilon_0 \epsilon_r \partial_{tt} E = -\frac{n^2}{c^2} \partial_{tt} E$$

$$\Delta \vec{E} - \frac{n^2}{c^2} \partial_{tt} \vec{E} = 0$$

Derivation - wave equation for $\vec{B}$ (taking curl of Ampere):

$$\nabla \times \nabla \times \vec{B} = \mu_0 \epsilon \partial_t \nabla \times \vec{E}$$

$$\nabla \nabla.\vec{B} - \Delta \vec{B} = -\mu_0 \epsilon \partial_{tt} \vec{B}$$

$$\Delta \vec{B} - \frac{n^2}{c^2} \partial_{tt} \vec{B} = 0$$

Plane-wave solutions:

$$\vec{E} = \vec{E}_0 \exp[i\vec{k}.\vec{r} - i\omega t + i\varepsilon] \quad , \quad \vec{B} = \vec{B}_0 \exp[i\vec{k}.\vec{r} - i\omega t + i\varepsilon]$$

dispersion relation (constraints propagation vector and frequency):

$$\vec{k}.\vec{k} = k^2 = \frac{\omega^2 n^2(\omega)}{c^2} \quad , \quad n(\omega)^2 = \epsilon_r$$

Phase (propagation) velocity and group velocity:

$$1/v_p = \frac{k(\omega)}{\omega} \quad , \quad 1/v_g = \frac{\partial k(\omega)}{\partial \omega} \quad , \quad v_g = \frac{v_p}{1 + \frac{\omega}{n}\frac{\partial n}{\partial \omega}}$$

transverse properties ($\vec{E}$, $\vec{B}$, and $\vec{k}$ form a right-handed system):

$$\vec{E}_0.\vec{k} = \vec{B}_0.\vec{k} = 0 \quad , \quad \vec{E}_0.\vec{B}_0 = 0 \qquad cB_0 = E_0 \quad , \quad \vec{B}_0 = \frac{1}{\omega}\vec{k} \times \vec{E}_0$$

in-vacuum relation between magnetic and electric intensities:

$$E_0 = Z_0 H_0 \quad \text{,impendance of free space : } Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\Omega$$

Energy density:

$$U = U_E + U_B = \frac{\epsilon}{2}\vec{E}.\vec{E} + \frac{1}{2\mu_0}\vec{B}.\vec{B} = \epsilon E^2$$

Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \quad , \quad S(\vec{r}, t) = \epsilon_0 n c E^2(\vec{r}, t)$$

Irradiance, time averaged energy flow:

$$< S >_T = I = \frac{1}{2}\epsilon_0 n c E_0^2$$

Photons

Photon energy, photon momentum

$$E_{ph} = \hbar\omega = h\nu \quad , \quad \hbar = h/(2\pi) \quad , \quad \hbar = 1.054 \times 10^{-34}\text{Js} \quad , \quad \vec{p} = \hbar\vec{k}$$

Photon flux density (particles per second through an area) and photon flux in a light beam with intensity I and power P:

$$\text{flux density} = \frac{I}{\hbar\omega} \quad , \quad \Phi = \frac{P}{\hbar\omega}$$

radiation pressure (total absorption, additional factor of 2 for total reflection):

$$\mathcal{P} = \frac{I}{\hbar\omega}\hbar|\vec{k}| = \frac{I}{c}$$

Gaussian beams:
Intensity profile (propagation along z), effective area, power:

$$I(r,z) = I_0 \left(\frac{w_0}{w(z)}\right)^2 e^{-2r^2/w(z)^2} \quad , \quad A = \frac{\pi w_0^2}{2} \quad , \quad P = AI_{max} = \frac{\pi w(z)^2}{2} I(r=0,z)$$

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \quad , \quad R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]$$

Gaussian beam waist (spot size): w_0
Rayleigh range z_R (confocal parameter $b = 2z_R$) and divergence angle Θ:

$$z_R = \frac{\pi w_0^2}{\lambda} \quad , \quad \Theta = \frac{2\lambda}{\pi w_0}$$

Interference
Young's two-slit experiment (hole-distnace h, observation screen distance x):

$$\frac{< S >_T}{\epsilon_0 n c} = \left(\frac{2A}{x}\right)^2 \frac{1}{2} \cos^2\left[\frac{kyh}{2x}\right]$$

fringe spacing:

$$\Delta y = \frac{\lambda x}{h}$$