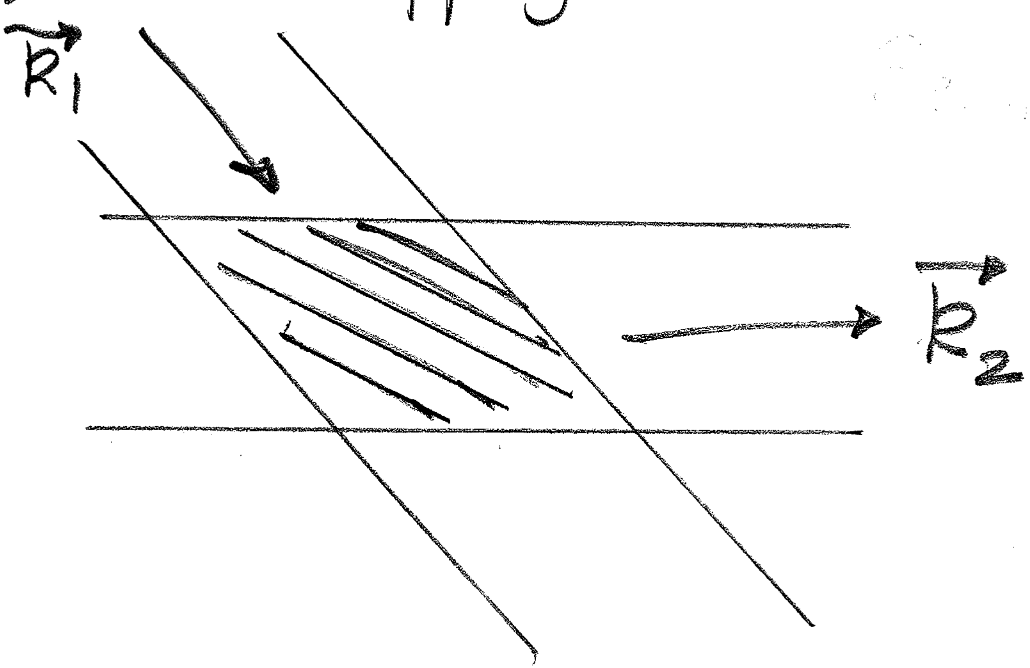
9. Interference

We saw in Sec. 5 that two optical fields that are not orthogonally polarized (Fresnel-Arago law) can produce interference using the two examples:

1) Two overlapping EM fields

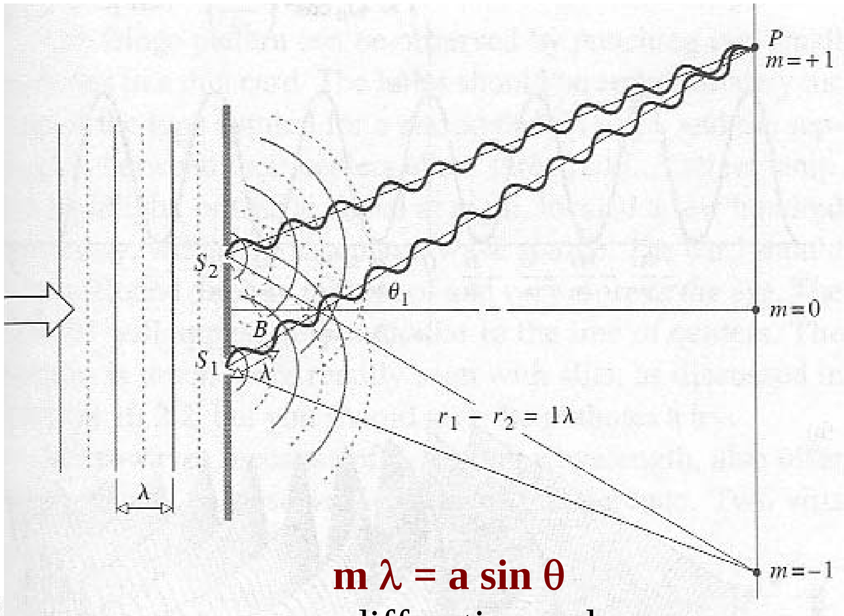


2) Youngs two-slit experiment, see next page.

Now we would like to broaden the discussion of interference using some specific and important examples.

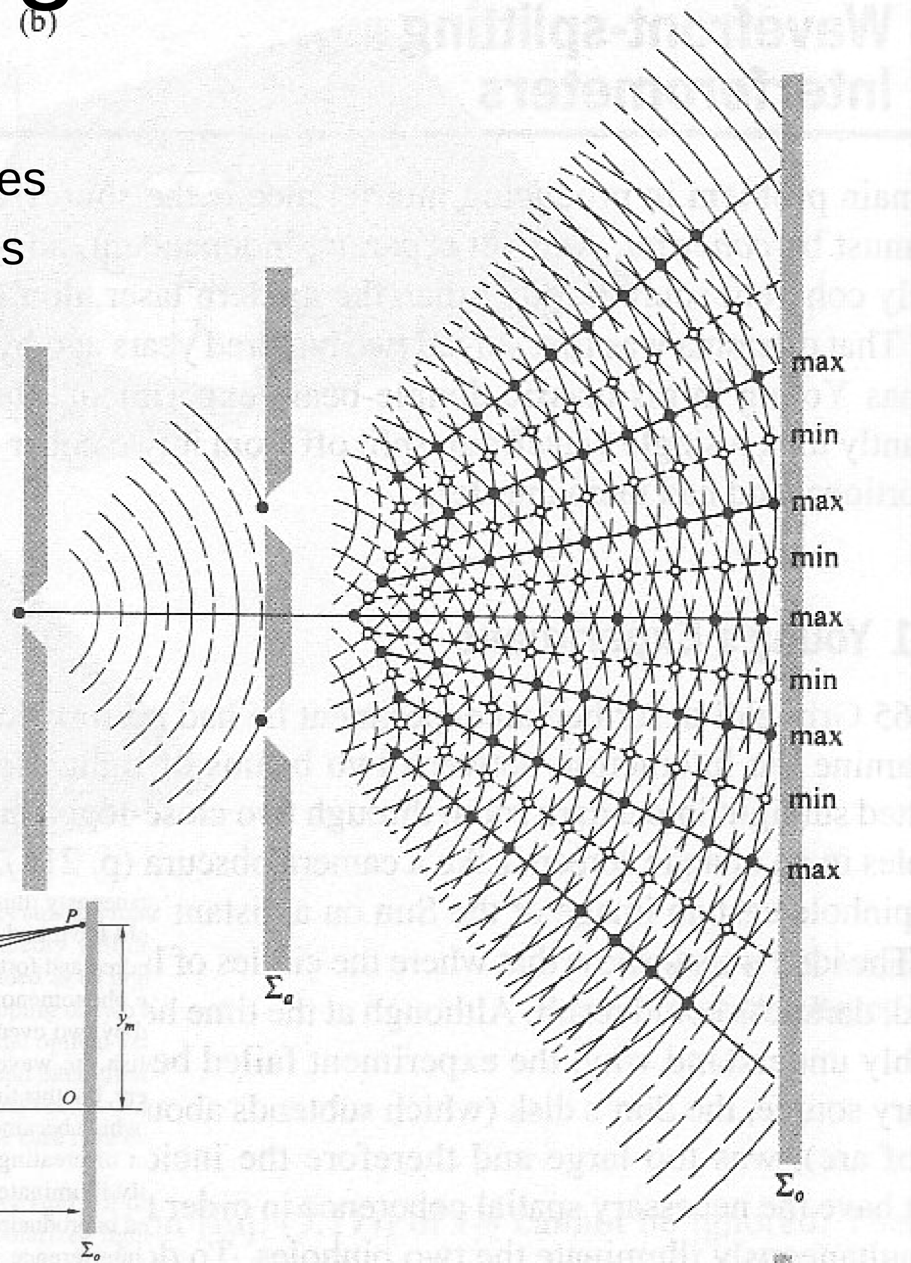
Wavefront-splitting interferometer

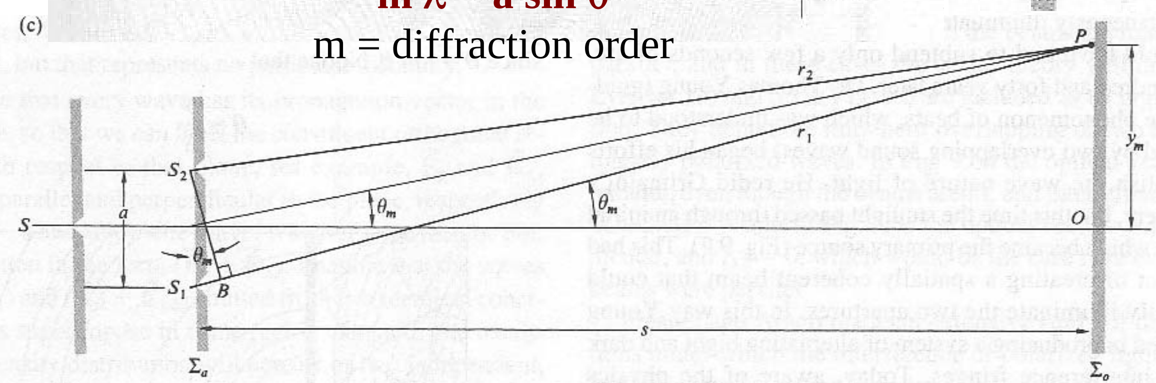
- Young's double slit experiment
  - Interference of two spherical waves
  - Equal path lengths -- linear fringes



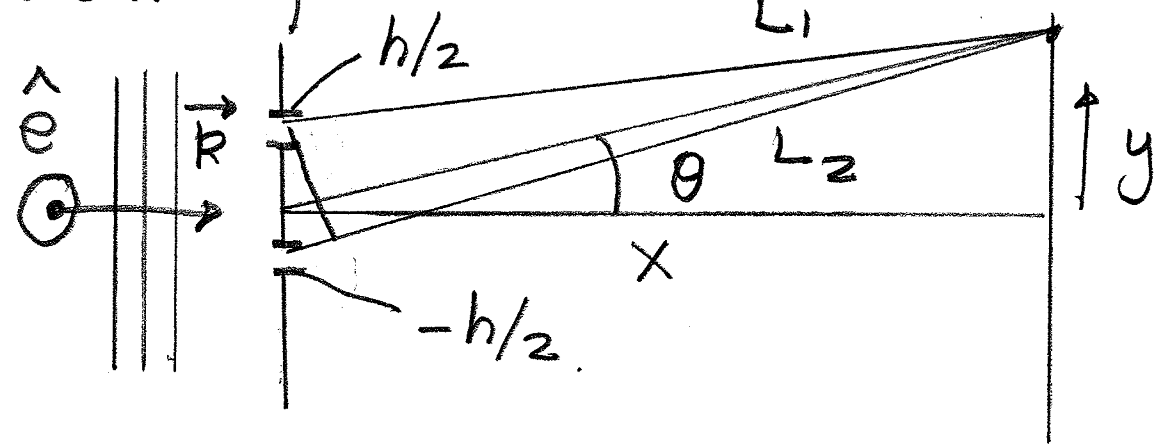
$m \lambda = a \sin \theta$

m = diffraction order





We start from the case of Young's two slit experiment.



This is an example of a <u>two-beam interferometer</u> based on <u>division of wavefront</u>.

Let's re-analyze Young's two slit expt.: Field polarized along z

$$E(y) = E_0 e^{ikL_1} + E_0 e^{ikL_2}$$

so

$$I \propto |E(y)|^2 \propto |E_0|^2 \left| e^{ikL_1} + e^{ikL_2} \right|^2$$

$$\propto \cos^2\left(\frac{k(L_2 - L_1)}{2}\right)$$

$k = \omega/c$, and $E_0 = (\mathcal{R}/x)$.

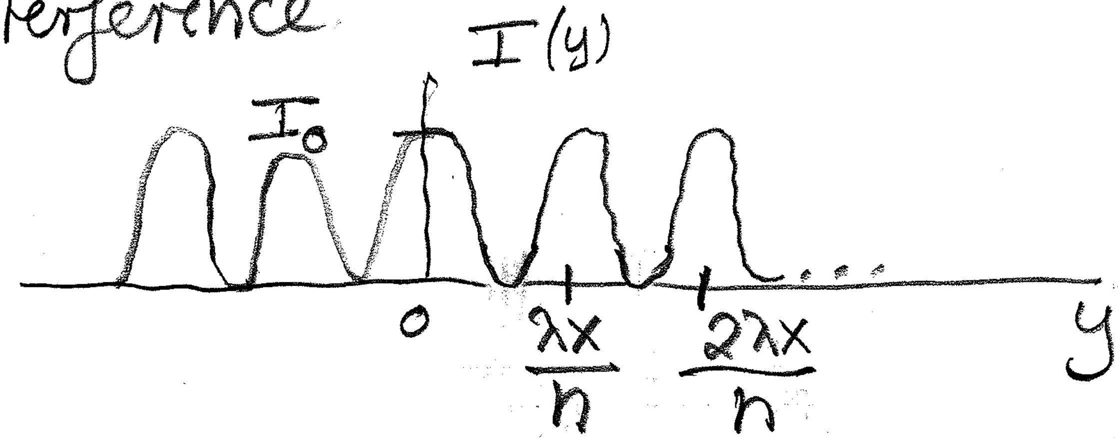
From the figure $(x \gg h, y)$

$$(L_2 - L_1) = h \sin\theta \sim h\left(\frac{y}{x}\right)$$

and finally

$$I(y) = I_0 \cos^2\left(\frac{kyh}{2x}\right)$$

which is of the same form as we found in Sec. 5, and shows the expected interference

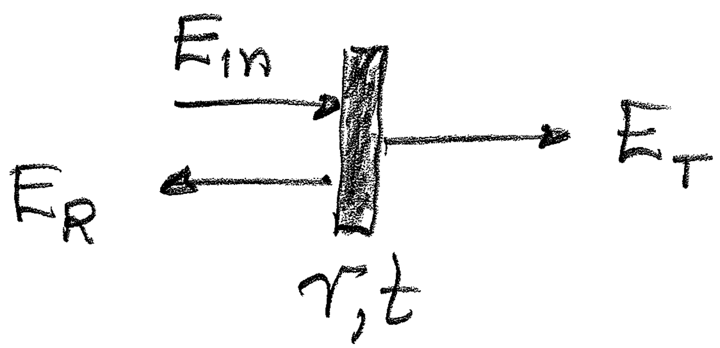


For a Young-type experiment with N-slits we have an <u>N-beam interferometer</u> based on division of <u>wave front</u>

Are there other types of two beam interferometer that involve combining two beams?

$$E = E_0 e^{ikL_1} + E_0 e^{ikL_2}$$

We shall consider two examples but first we consider an optical element called a beam splitter (or mirror)



r & t field reflection & trans. coefficients (complex)

then

$$\boxed{E_R = r E_{in}, \quad E_T = t E_{in}}$$

For a lossless beam splitter

$$I_{in} = I_R + I_T, \quad \text{or}$$

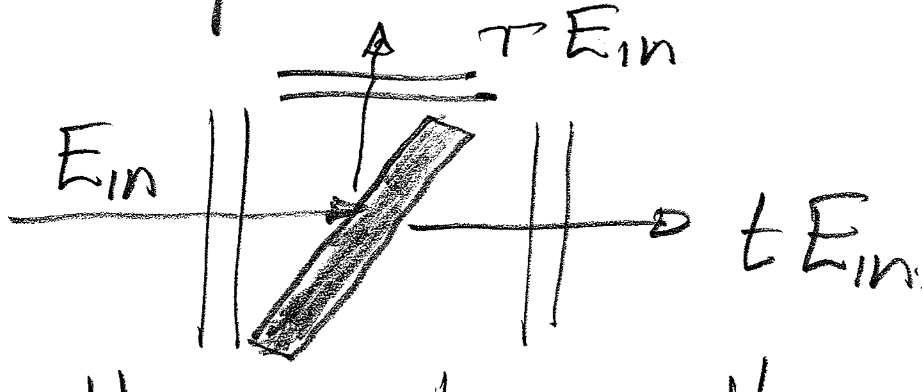
$$|E_{in}|^2 = |E_R|^2 + |E_T|^2$$

which implies the input power equals the sum of reflected and transmitted powers. We then obtain

$$|E_{in}|^2 = |r|^2 |E_{in}|^2 + |t|^2 |E_{in}|^2$$

or $\boxed{1 = R + T}$ (so if $R=1 \Rightarrow T=0$ and if $T=1$, $R=0$)

where $R = |r|^2$, $T = |t|^2$ are the intensity reflection and transmission of the mirror or beam splitter. (BS). Note the a BS <u>divides the amplitude</u> of the input wave



but otherwise leaves the wavefront intact - as opposed to a slit which truncates the wavefront.

We next consider two other examples of 2-beam interferometers but now based on <u>division of amplitude</u>, or amplitude splitting interferometers.

- <u>Michelson interferometer</u> (see figure)

$r = 1/\sqrt{2}$, $t = i/\sqrt{2}$, $R + T = 1$

$$E = E_{in} r t e^{ikL_1} + E_{in} r t e^{ikL_2}$$

$$\longrightarrow |E|^2 = |E_{in}|^2 \cos^2\left(\frac{k(L_2 - L_1)}{2}\right)$$

- <u>Mach-Zehnder</u> (see figure)

$$E_1 = r t E_{in} e^{ikL_1} + r t E_{in} e^{ikL_2}$$

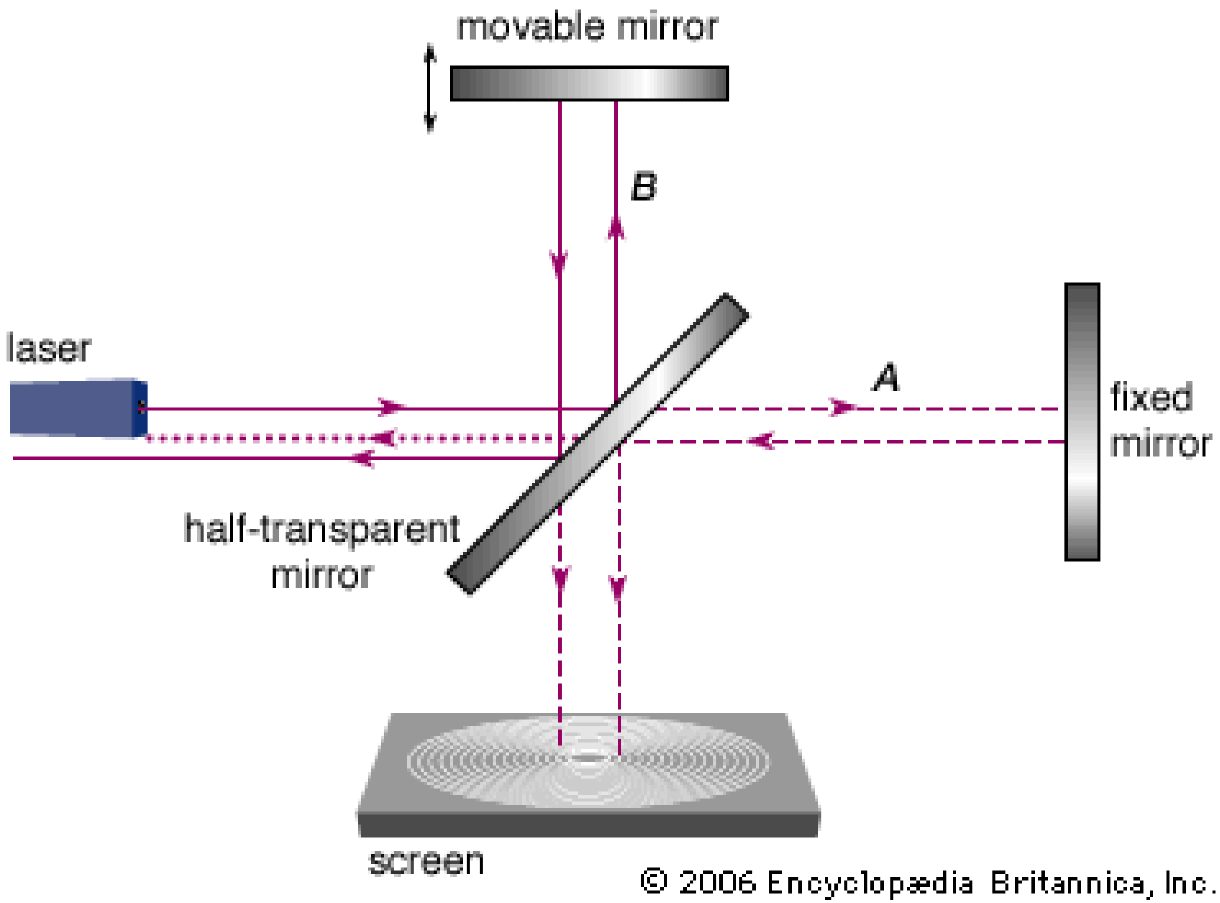
$$E_2 = r^2 E_{in} e^{ikL_2} + t^2 E_{in} e^{ikL_2}$$

$$|E_1|^2 = |E_{in}|^2 \cos^2\left(\frac{k(L_2 - L_1)}{2}\right)$$

$$|E_2|^2 = |E_{in}|^2 \sin^2\left(\frac{k(L_2 - L_1)}{2}\right)$$

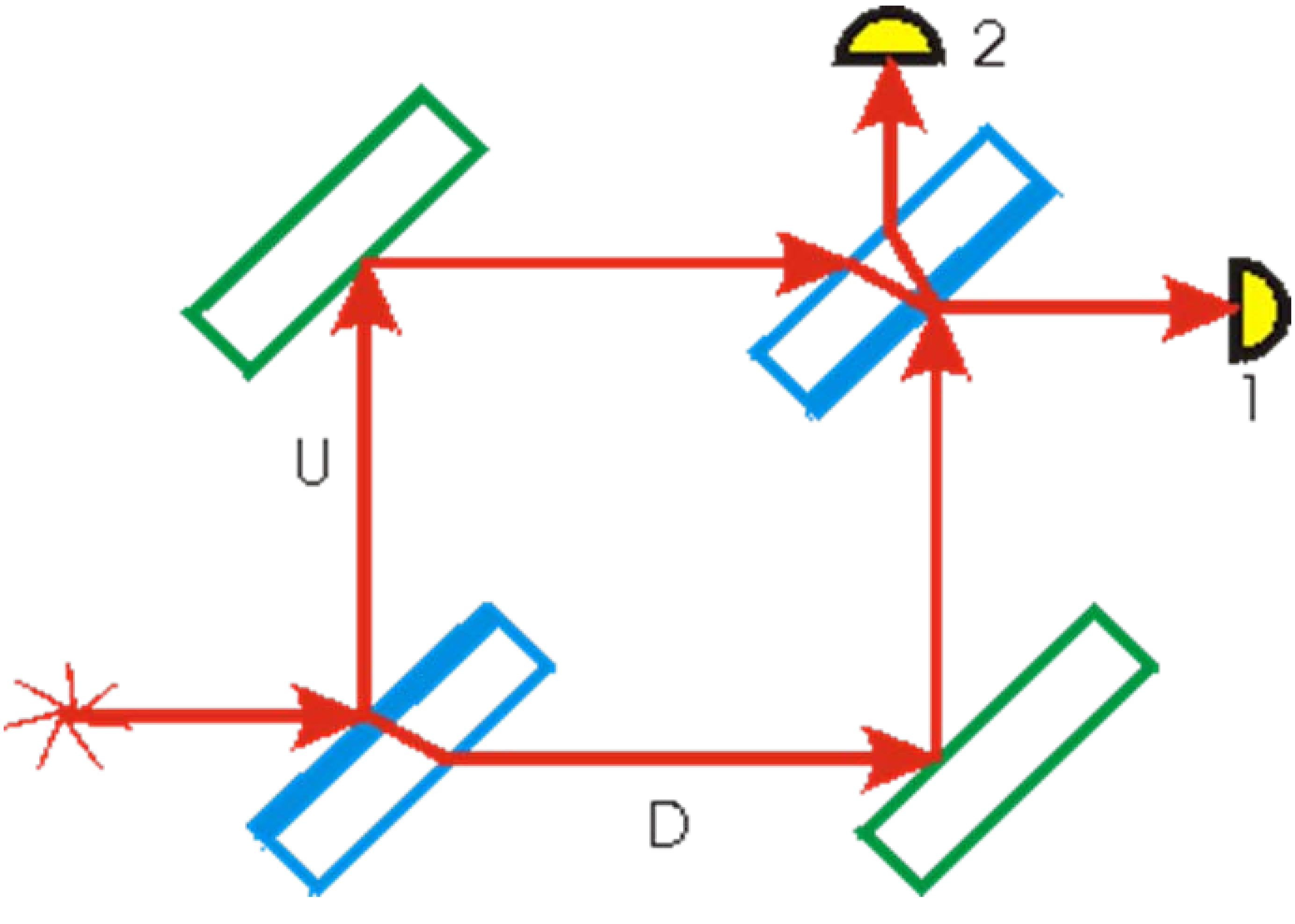
Amplitude-splitting interferometers

- *Michelson Interferometer*

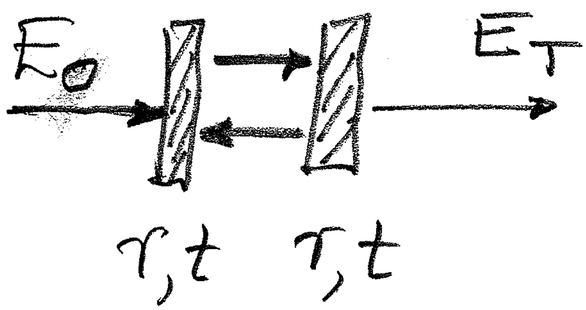


Amplitude-splitting interferometers

- *Mach-Zehnder interferometer*



Next we look at the Fabry-Perot which is a <u>multi-beam interferometer</u> based on division of amplitude. It is composed of two parallel mirrors



since the first mirror reduces E_0 to tE_0, it seems clear $E_T = t^2 E_0$ and

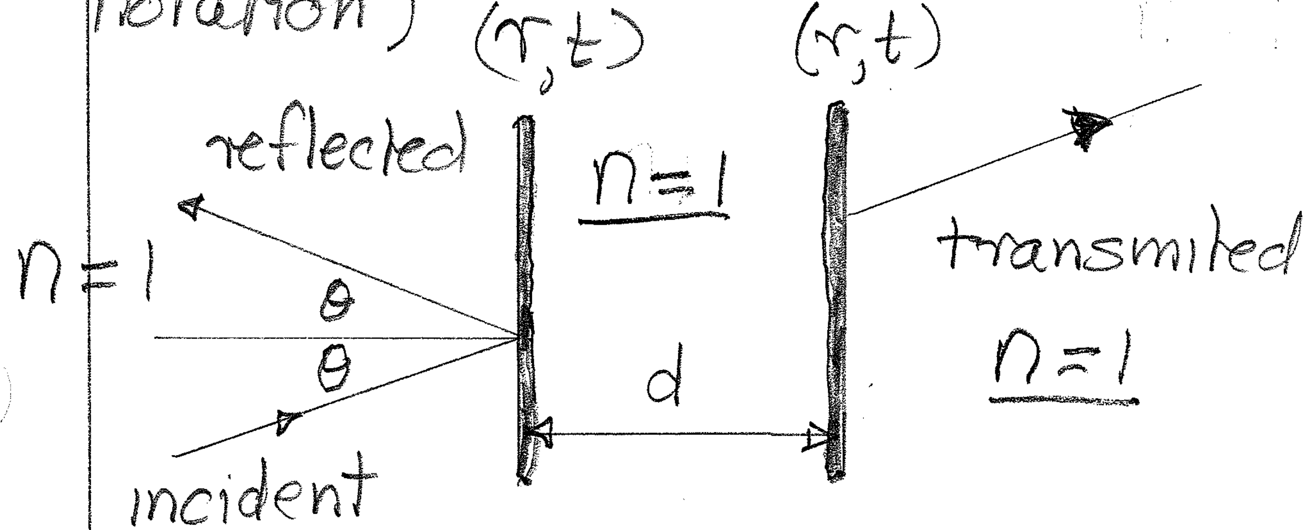
$$|E_T|^2 = T^2 |E_0|^2$$

$$= (1-R)^2 |E_0|^2$$

But this is not the case and it can happen that $|E_T|^2 = |E_0|^2$ even for $R \simeq 1$, $T \simeq 0$! Read on...

Fabry-Perot interferometer.

This is covered in Fowles 4.1-4.3 and a bit more generally in Hecht 9.6. The basic Fabry-Perot (FP) interferometer or etalon is shown below (Fowles notation)



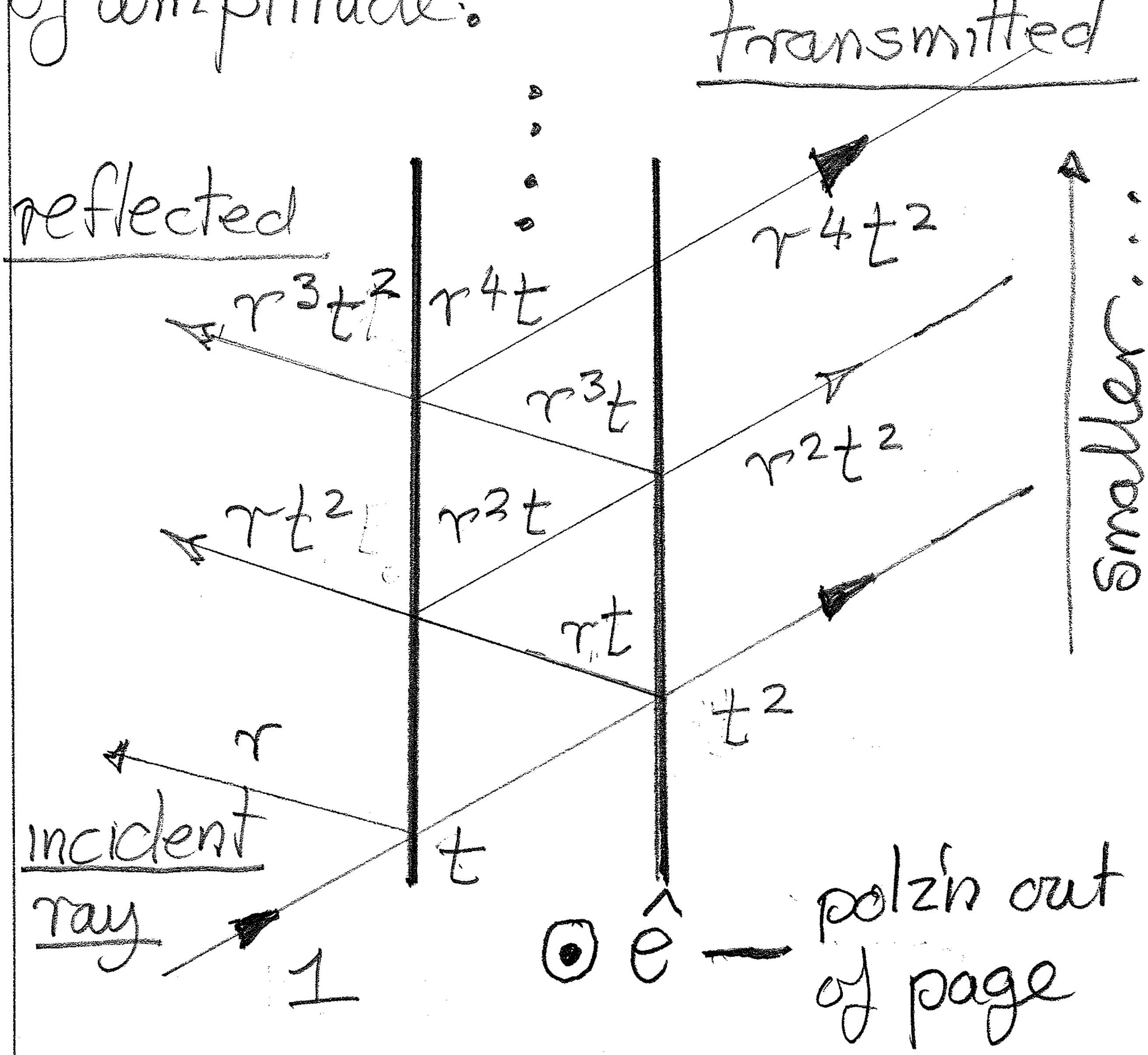
Incident harmonic wave of frequency ω at angle θ incident on pair of identical reflectors, or mirrors, of amplitude reflection coefficient r and transmission coefficient t

$$|r|^2 + |t|^2 = R + T = 1$$

(intensity reflectivities)

that is, the mirrors are lossless.
(We further assume r & t are not frequency dependent, and later we shall show how such mirrors can be realized using thin film multi-layers).

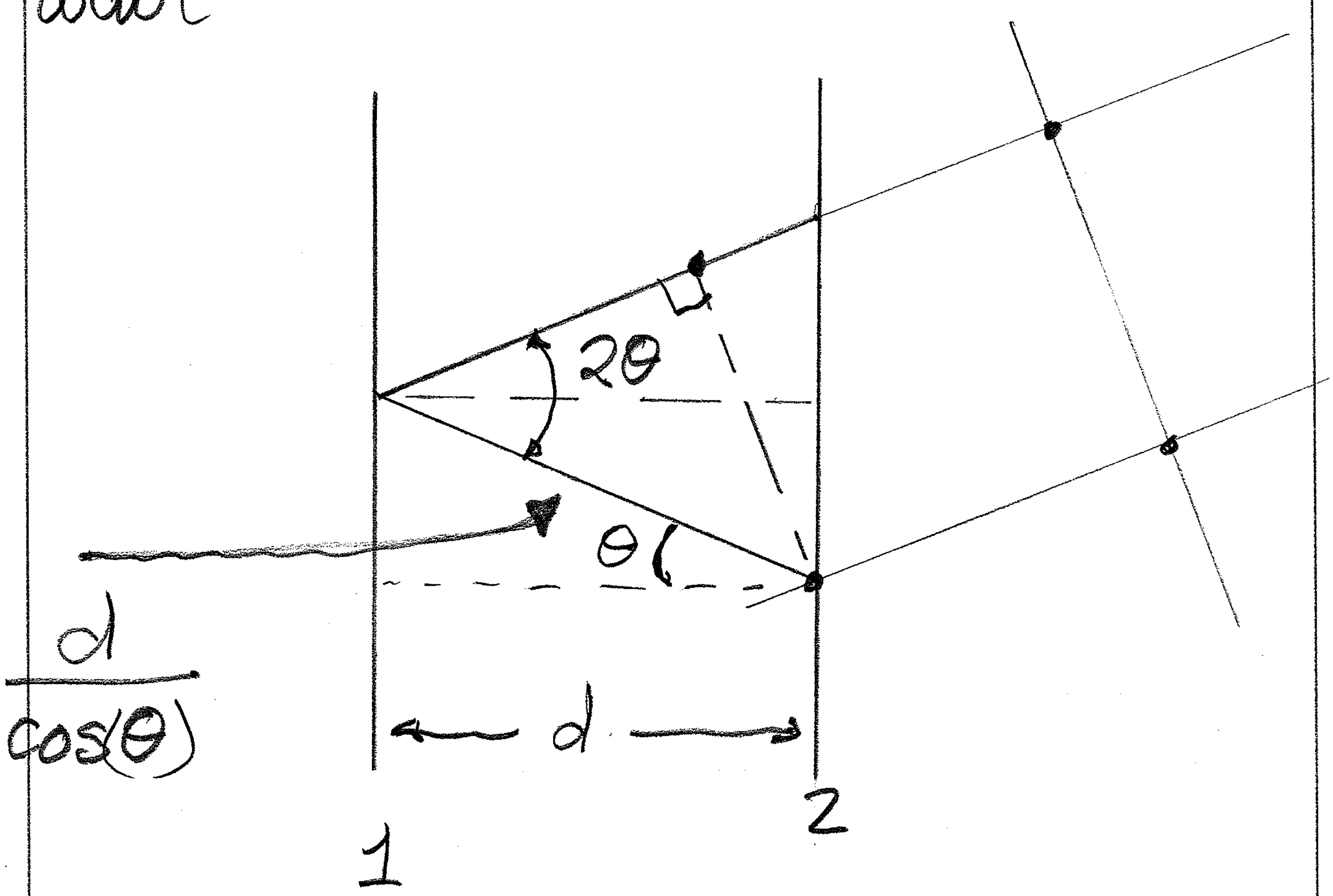
The FP is a multi-interference optical device based on division of amplitude:



To get the total transmitted field due to an input field E_0 (assume field is polarized out of the page) then we need to sum the partial waves

$$E_T = E_0 t^2 + \underbrace{E_0 t^2 r^2 + E_0 t^2 r^4 \ldots}_{\longrightarrow \text{smaller}}$$

But there is also a phase shift between partial waves due to the $\exp(i\vec{k}\cdot\vec{r})$ factor in the harmonic wave



Phase factor goes as

$$\sim \exp(ikL), \quad L \text{ distance.}$$

Path difference between adjacent rays (partial waves) is

$$L = \frac{d}{\cos(\theta)} + \frac{d}{\cos(\theta)} \cos(2\theta)$$

$$= \frac{d}{\cos(\theta)} \left(1 + \cos(2\theta)\right) = 2\cos^2\theta$$

$$= 2d\cos\theta$$

This gives a phase factor

$$e^{ikL} = e^{i\delta}, \quad \boxed{\delta = 2kd\cos\theta}$$

or

$$\boxed{\begin{aligned} \delta &= \frac{4\pi d}{\lambda} \cos\theta \\ &= 2\left(\frac{\omega}{c}\right) d\cos\theta \end{aligned}}$$

The transmitted field becomes

$$E_T = E_0 t^2 + E_0 t^2 r^2 e^{i\delta} + E_0 t^2 r^4 e^{2i\delta} + \ldots$$

$$= E_0 t^2 \left(1 + (r^2 e^{i\delta}) + (r^2 e^{i\delta})^2 + \ldots\right)$$

$$= E_0 t^2 (1 + \xi + \xi^2 + \xi^3 + \ldots)$$

This is a geometric series which can be summed ($\xi = r^2 e^{i\delta}$), $|\xi| < 1$

$$(1 + \xi + \xi^2 + \ldots) = \frac{1}{(1 - \xi)}$$

Thus, we obtain

$$\boxed{E_T = \frac{t^2 E_0}{(1 - r^2 e^{i\delta})}}$$

which contains all orders of multi-reflection! The incident intensity is $I_0 = \frac{1}{2} c\epsilon_0 |E_0|^2$, and the transmitted intensity $I_T = \frac{1}{2} \epsilon_0 c |E_T|^2$.

We then obtain

phase upon reflection.

$$I_T = I_0 \frac{|t|^4}{|1 - r^2 e^{i\delta}|^2}$$

Now $|t|^4 = T^2$, and write $r = |r| e^{i\delta_r/2}$ $= \sqrt{R}\, e^{i\delta_r/2}$. Then

$$I_T = I_0 \frac{T^2}{(1 + R^2 - 2R\cos\Delta)}$$

with $\Delta = \delta + \delta_r$. This can be rearranged (see p. 88 of Fowles) into the form

$$\frac{I_T}{I_0} = \frac{1}{1 + F\sin^2(\Delta/2)} = \mathcal{T}$$

which is the Airy function after George Airy, with

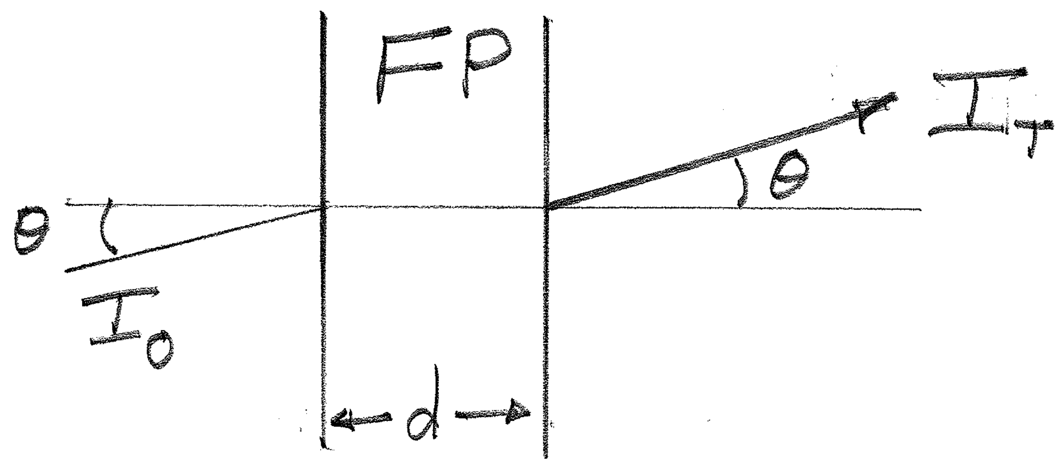
$$F = \frac{4R}{(1-R)^2},$$ coefficient of finesse

and the phase shift (difference)

$$\Delta = \delta_r + \frac{4\pi d}{\lambda}\cos\theta$$

incident angle

reflection   round trip

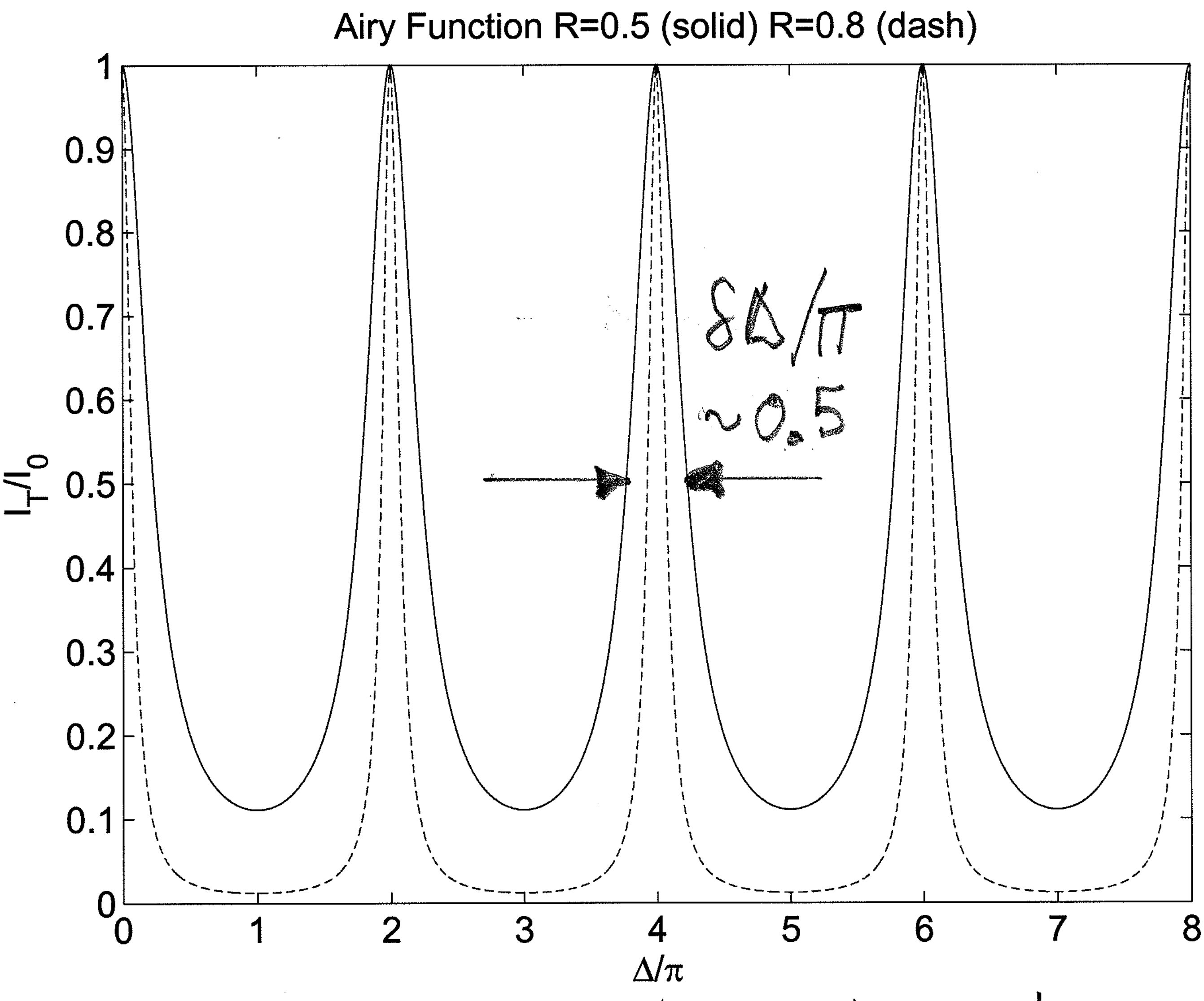


The Airy function gives the FP transmission $(I_T/I_0) = 1/(1+F\sin^2(\frac{\Delta}{2}))$

- Unity for $\Delta = 2\pi N$, $N = 0, \pm1, \pm2$, N is the order of interference – this is FP resonance
- Minimum value $\left(\frac{1}{1+F}\right)$ for $\Delta = (2m+1)\pi$, $m = 0, \pm1, \pm2..$ off-resonance
- FP etalon can be perfectly transmitting even with high reflecting mirrors!!

FP resonances narrower as $R \to 1$, $T \to 0$, low loss case, $F = 4R/(1-R)^2 >> 1$, high finesse FP.

Minimum off-resonance $= \dfrac{1}{1+F}$



R	F	Min	Max	$\mathcal{F}$
0.5	8	0.11	1	4.44
0.8	80	0.012	1	14.0

off-resonance

resonance.

The width of the Airy function peaks may be evaluated as follows for the high finesse case:

$$(I_T/I_0) = \frac{1}{1+F\sin^2(\Delta/2)}, \quad F \gg 1$$

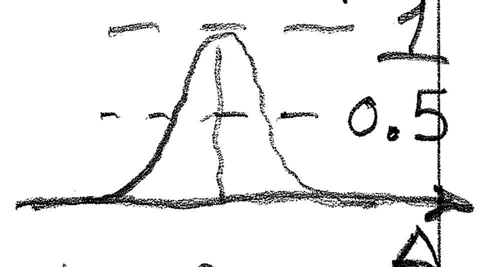
This is unity for $\Delta_N = 2\pi N$. Set $\Delta = \Delta_N + \eta$, η a <u>small deviation</u>

$$\sin\left(\frac{\Delta}{2}\right) = \sin\left(\frac{\Delta_N}{2} + \frac{\eta}{2}\right) = \sin(N\pi + \eta/2)$$

$$= \pm \sin(\eta/2) \simeq \pm \eta/2, \text{ small } \eta$$

then

$$(I_T/I_0) \sim \frac{1}{(1+F\eta^2/4)}$$



$= 1$ for $\eta = 0$ (resonance). This falls to $(1/2)$ for $\eta = \pm 2/\sqrt{F}$, so the FWHM of the fringes is

$$\boxed{\delta\Delta = \frac{4}{\sqrt{F}} = \frac{2(1-R)}{\sqrt{R}} = \frac{2\pi}{\mathcal{F}}}$$

$\mathcal{F} = \pi\sqrt{R}/(1-R)$ — <u>finesse</u>.

Since the mirrors are lossless energy conservation demands

$$I_0 = I_T + I_R$$

reflected intensity

transmitted

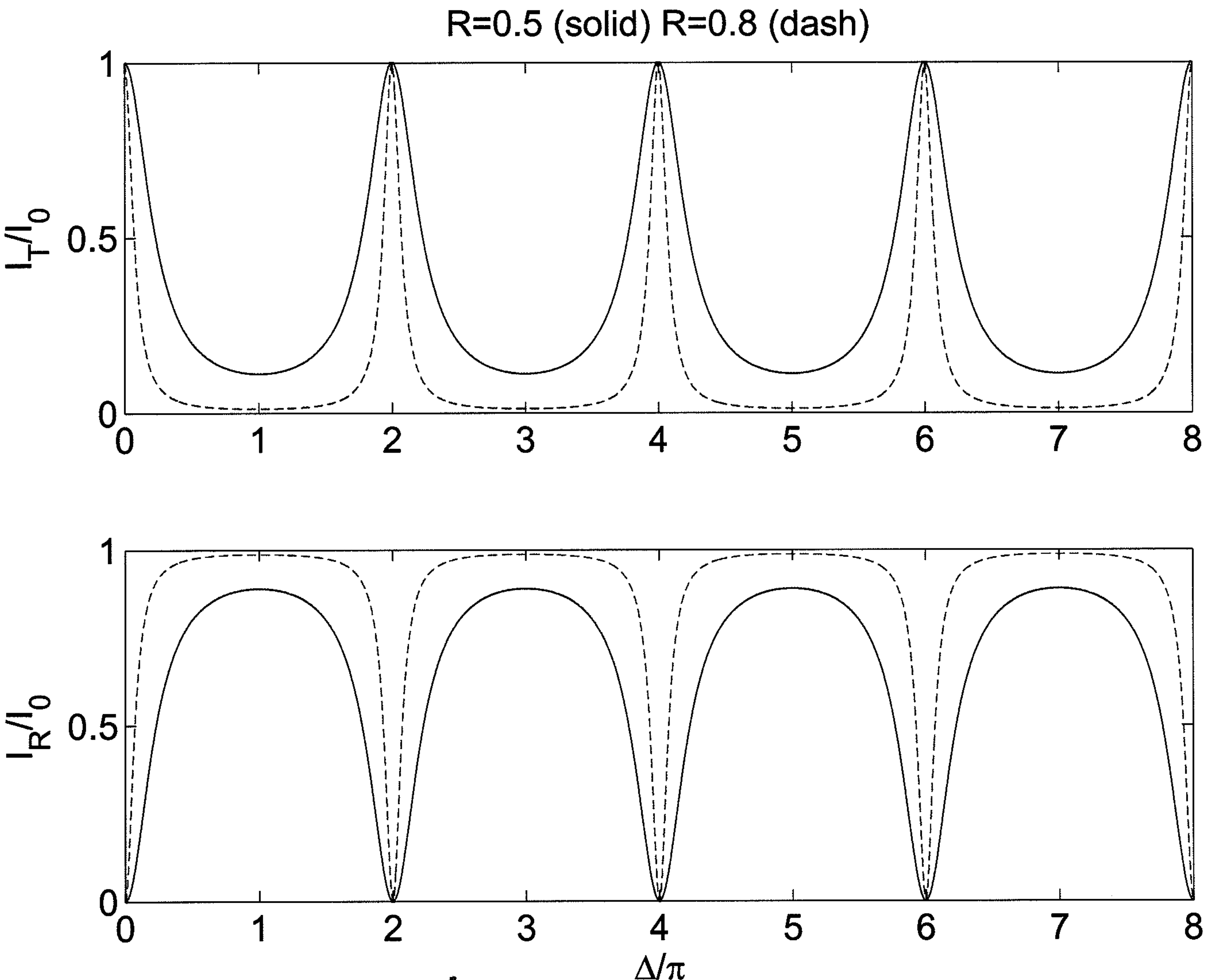
or

$$\frac{I_R}{I_0} = 1 - \frac{I_T}{I_0} = \frac{F\sin^2(\Delta/2)}{1+F\sin^2(\Delta/2)} = \mathcal{R}$$

Notes on FP reflection

- zero for $\Delta = 2\pi N$, $N = 0, \pm 1, \pm 2 \ldots$
- Maximum $\left(\frac{F}{1+F}\right)$ for $\Delta = (2m+1)\pi$.
- Thus, for a high finesse FP $F \gg 1$, $R \to 1$, the max reflectivity $\to 1$ off resonance.
- The minumum reflectivity is zero on-resonance when the transmission goes to unity, even for high reflecting mirrors!!

Transmission



Reflection

Fringes of equal inclination.

So far we have looked at the transmission & reflection of the FP as a function of Δ

$$\Delta = \delta_r + \frac{4\pi d}{\lambda}\cos\theta.$$

There are a number of variables

- d – mirror spacing
- λ – incident wavelength
- θ – incident angle.

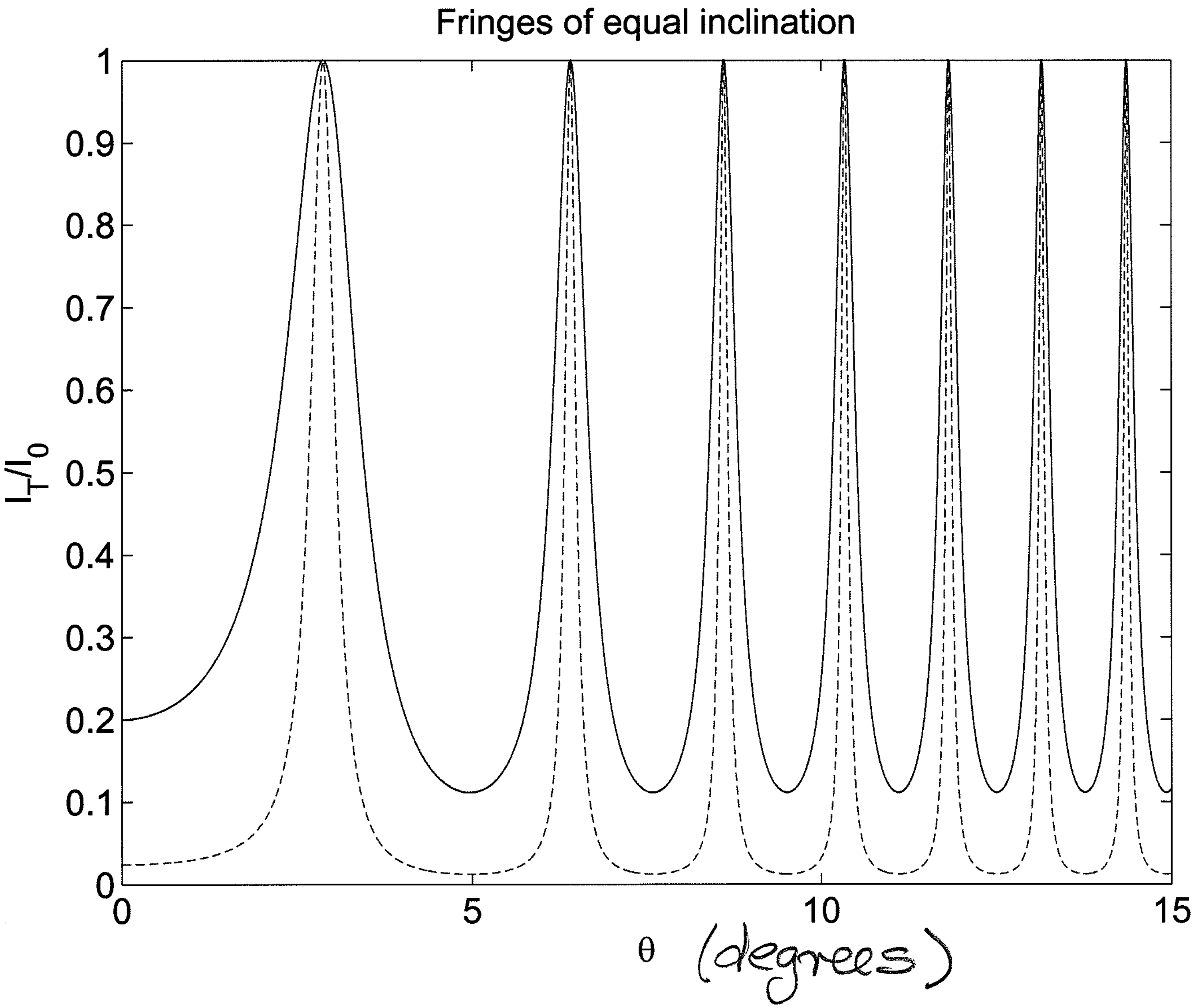
etalon

Consider the case that d & λ are fixed but θ is variable, then

$$\frac{I_T}{I_0} = \frac{1}{1+F\sin^2(\Delta/2)}$$

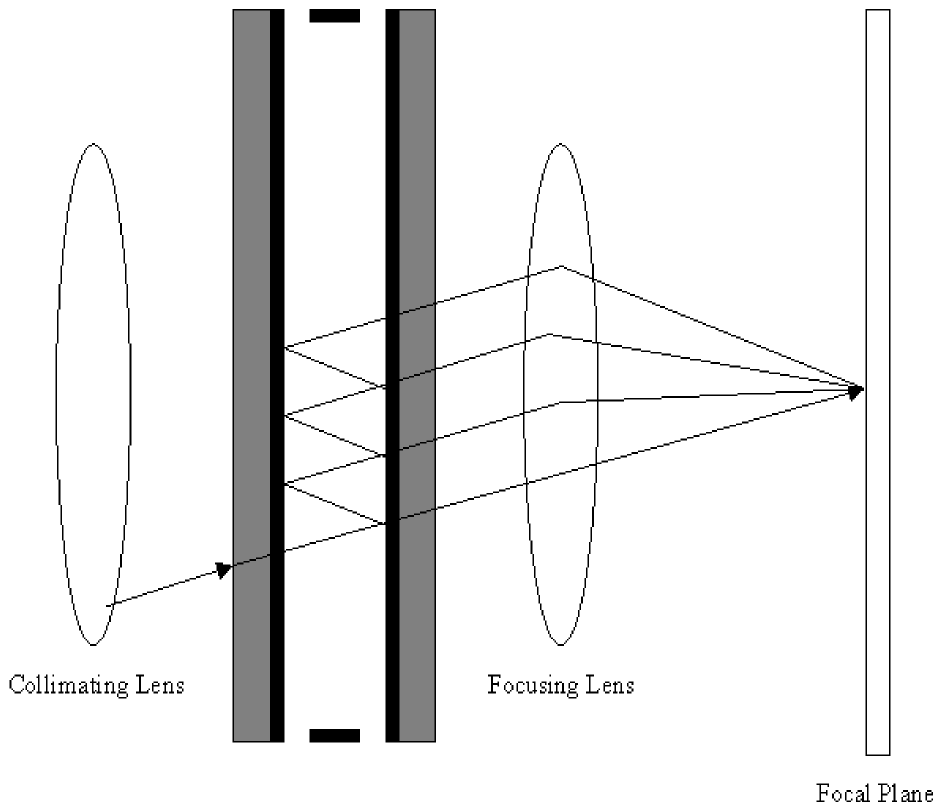
→ fringes of equal inclination

$R = 0.5$ (solid) $R = 0.8$ (dash)
$\lambda = 1\mu m$, $d = 100\mu m$, $\delta_r = \pi/2$

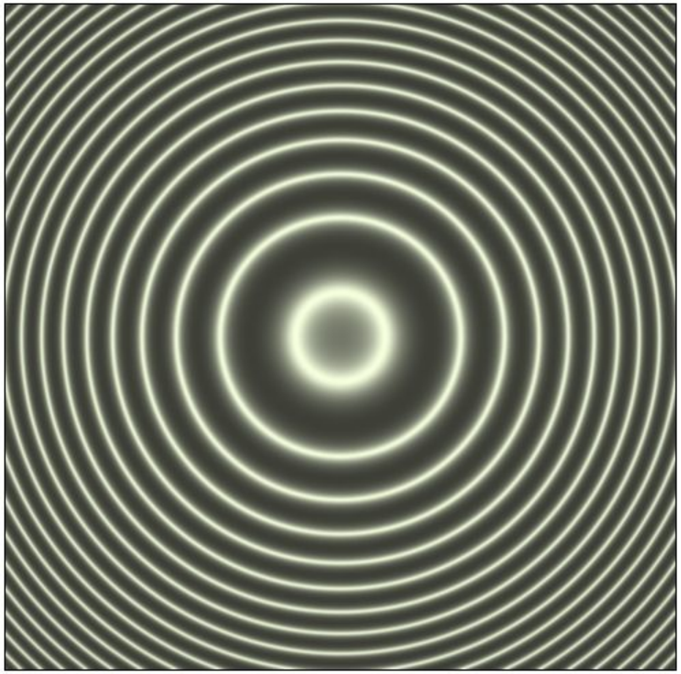


Fringes of equal inclination

Experimental arrangement

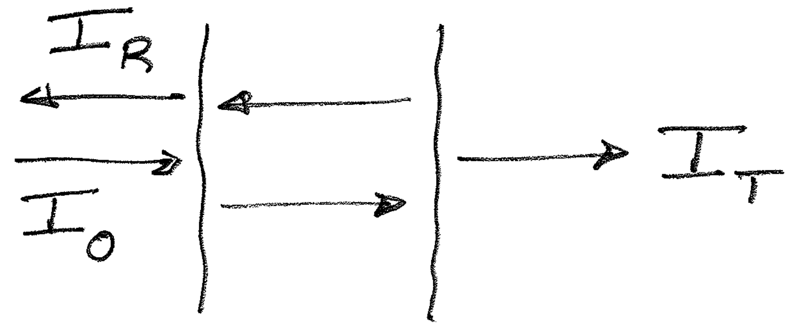


Far field fringes



Normal incidence

Hereafter we consider the case of normal incidence $\theta = 0$



$$\left(I_T / I_0\right) = \frac{1}{1 + F\sin^2(\Delta/2)} = 1 - \frac{I_R}{I_0}$$

$$\Delta = \delta_r + \frac{4\pi d}{\lambda} = \delta_r + 2kd$$

$$= \delta_r + \frac{2\omega d}{c} = \delta_r + \frac{4\pi \nu d}{c}$$

We note Δ is the phase-shift for a round trip of length $2d$, δ_r is phase from reflections.

So we can vary transmission $\mathcal{T}$ for fixed d, vary λ (or ω or ν)

$\longrightarrow$ etalon

or we can vary (I_T/I_0) for fixed ν by varying $d \rightarrow$ interferometer

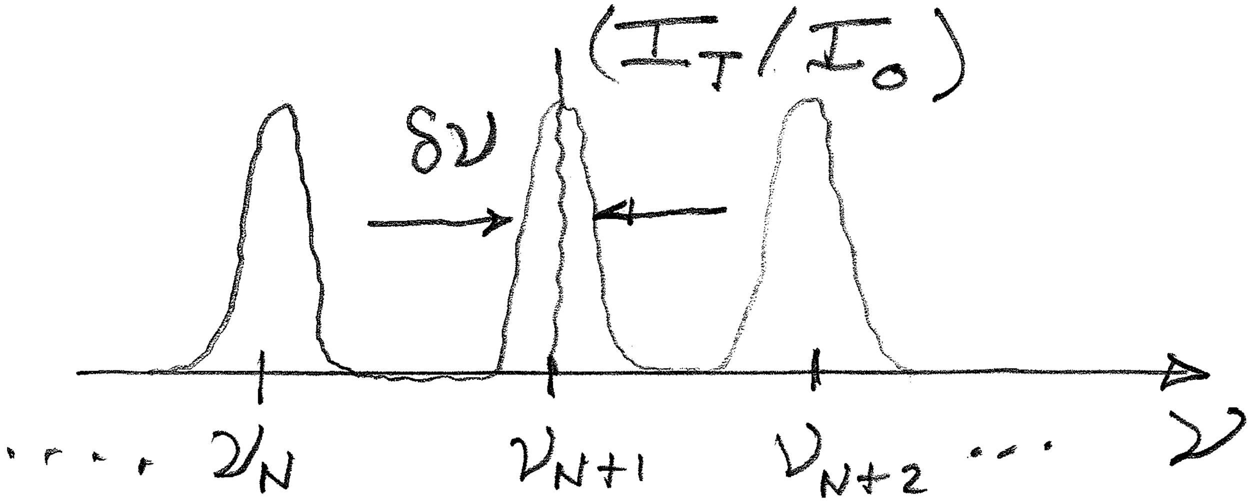
Example: Set $\delta_r = 0$, then for fixed d the FP transmission is maximum for (on-resonance)

$$\Delta_N = \frac{4\pi \nu_N d}{c} = 2\pi N$$

or

$$\nu_N = \left(\frac{c}{2d}\right) N$$

order of interference



Could also plot vs $\omega = 2\pi\nu_N$,

$\lambda_N = c/\nu_N$ ....

How narrow will the FP fringes be in this form? We have

$$\Delta_N = \frac{4\pi \nu_N d}{c}$$

$$\Delta_N + \delta\Delta = \frac{4\pi(\nu_N + \delta\nu)d}{c}$$

$\delta\Delta = 2(1-R)/\sqrt{R} = 2\pi/\mathcal{F}$. So FWHM of FP fringes $\delta\nu$ is.

$$\boxed{\delta\nu = \frac{1}{\mathcal{F}}\left(\frac{c}{2d}\right)}$$

$(c/2d)$ – free spectral range ν_F spacing between adjacent FP resonances

$$\boxed{\left(\frac{c}{2d}\right) = \nu_{N+1} - \nu_N}$$

$\mathcal{F} = \pi\sqrt{R}/(1-R)$, finesse.

The FP resonances are given by

$$\nu_N = N\left(\frac{c}{2d}\right) = \frac{\omega_N}{2\pi}$$

and the FWHM of the resonances

$$\delta\nu = \frac{\delta\omega}{2\pi} = \frac{1}{\mathcal{F}}\left(\frac{c}{2d}\right) = \frac{\nu_N}{N\mathcal{F}}$$

ω_N & $\delta\omega$ being the equivalent in terms of the angular temporal frequency.

We know $\nu = c/\lambda$, so

$$\delta\nu = -\frac{c}{\lambda^2}\delta\lambda = -\nu\left(\frac{\delta\lambda}{\lambda}\right)$$

and $|\delta\nu/\nu| = |\delta\lambda/\lambda| = |\delta\omega/\omega|$.

Using $\nu = N(c/2d)$, we get

$$\frac{\delta\nu}{\nu} = \frac{\delta\lambda}{\lambda} = \frac{\delta\omega}{\omega} = \frac{1}{N\mathcal{F}}$$

N - order of interference

$\mathcal{F}$ - Finesse.

Longitudinal modes

The FP transmission is maximum when the round trip phase-shift

$$\Delta_N = \frac{4\pi \nu_N d}{c} + \delta_r = 2\pi N$$

then the FP is on resonance. Thus in general

$$\nu_N = \left(\frac{c}{2d}\right)N - \underbrace{\frac{\delta_r c}{4\pi d}}_{\text{shift}}$$

these are the frequencies of the longitudinal modes of the FP, and we find

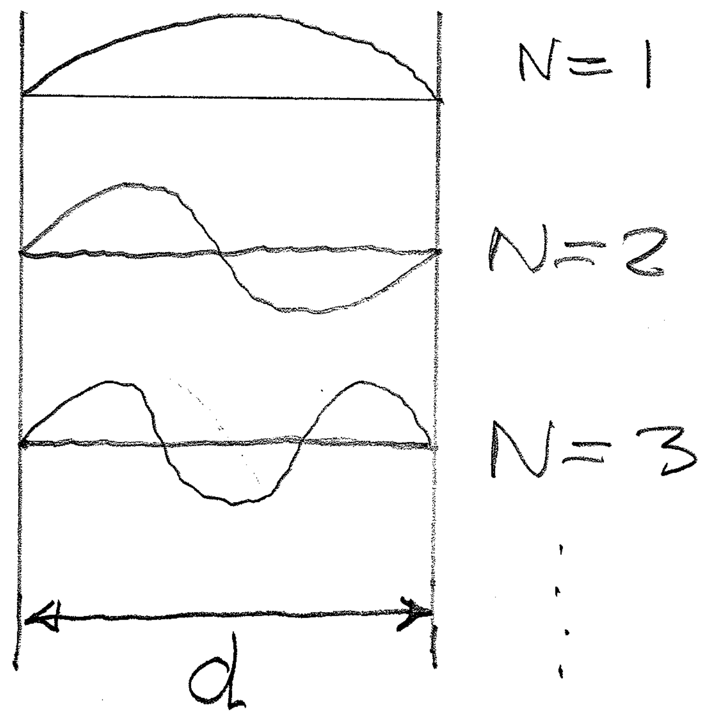
$$\nu_{N+1} - \nu_N = \underbrace{\left(\frac{c}{2d}\right)}_{\text{free spectral range}}$$

$(c/2d)$ is the temporal frequency difference between adjacent longitudinal modes.

Example: Set $\delta_r = 0$, then

$$\Delta_N = \frac{4\pi d}{\lambda_N} = 2\pi N$$

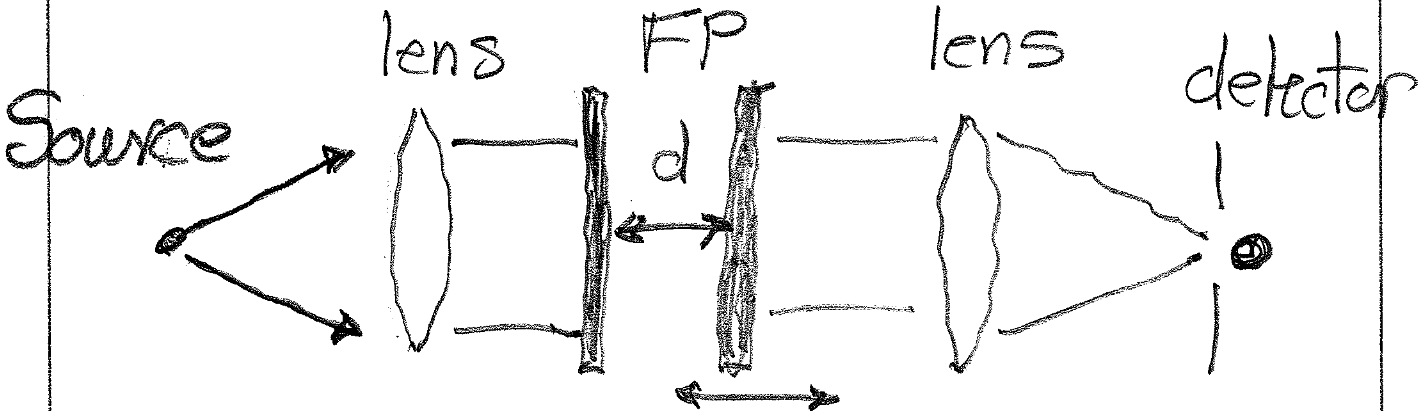
$$\lambda_N = 2d/N$$



Longitudanal modes correspond to wavelength that fit into box of length d.. Relevant concept for lasers, laser cavity length sets laser mode frequencies

Scanning FP interferometer

Here the geometry is



Here one scans the FP length d, scanning method. As before

$$\left(\frac{I_T}{I_0}\right) = \frac{1}{1+F\sin^2(\Delta/2)}$$

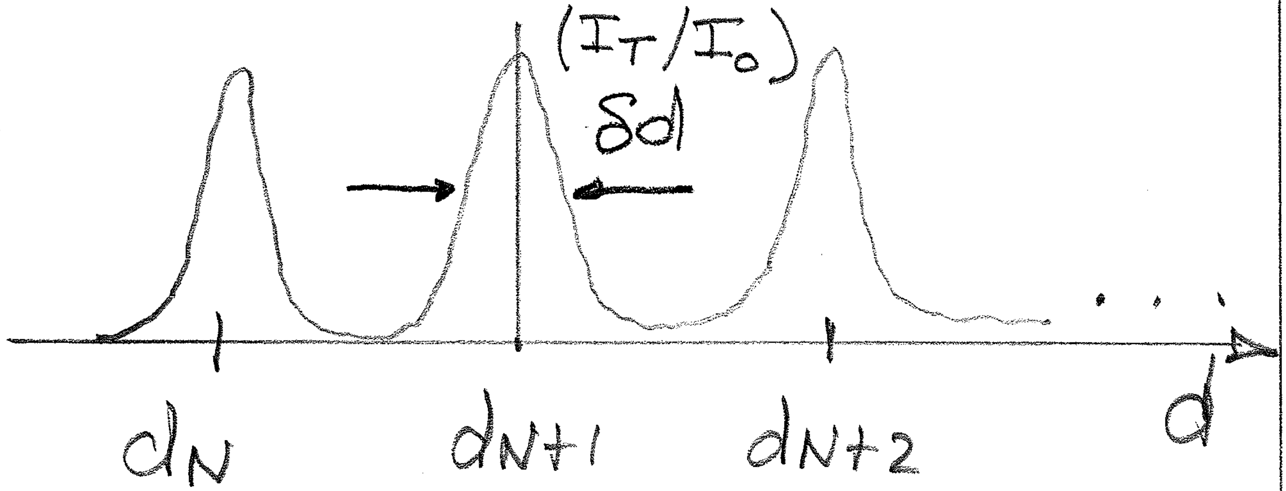
$$\Delta = \delta_r + 2kd = \delta_r + \frac{4\pi d}{\lambda} = \delta_r + \frac{4\pi\nu d}{c}$$

Setting $\delta_r = 0$, if the source has wavelength λ the FP transmission will have peaks at values of FP separation d_N

$$\Delta_N = 4\pi d_N / \lambda = 2\pi N$$

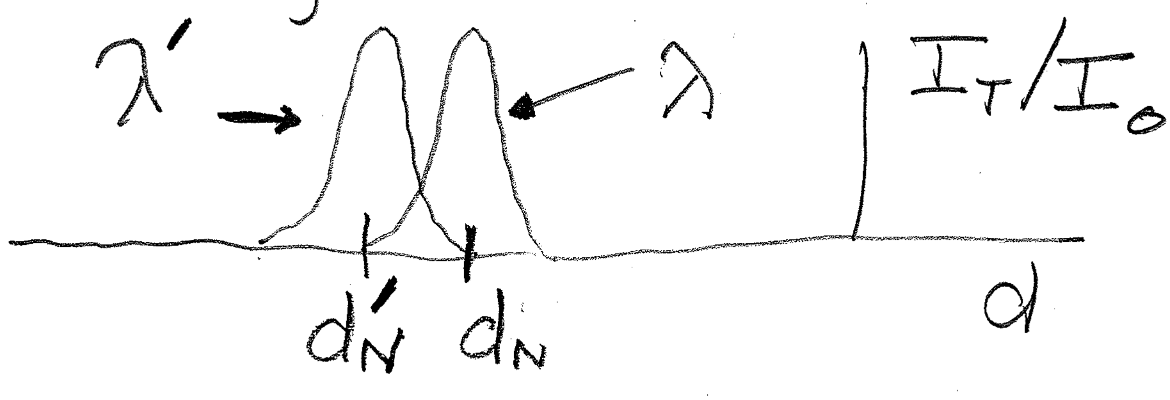
or

$$\boxed{d_N = N(\lambda/2).}$$



This can be used to measure λ for a source by scanning d.

What if there is more than one wavelength in the soure? We get a combination of Airy function, one for each λ



Can we resolve these two wavelengths?
We have ($\lambda \gtrsim \lambda'$)

$$d_N = N(\lambda/2)\ ,\ d_N' = N(\lambda'/2)$$

so

$$(d_N - d_N') = N(\lambda-\lambda')/2.$$

The FWHM of the FP resonance is given by $\delta d/d_N \sim 1/N\mathcal{F}$. In order to resolve the wavelengths we need

$$(d_N - d_N') > \delta d = d_N/N\mathcal{F} = \left(\frac{\lambda}{2}\right)\frac{1}{\mathcal{F}}.$$

or

$$N\left(\frac{\lambda-\lambda'}{2}\right) > \left(\frac{\lambda}{2}\right)\frac{1}{\mathcal{F}}$$

or

$$\frac{\delta\lambda}{\lambda} = \frac{(\lambda-\lambda')}{\lambda} > \frac{1}{N\mathcal{F}}$$

This is the Rayleigh or Taylor criterion (p. 96 Fowles). Thus, wavelength resolution increases as N and $\mathcal{F}$ get larger.

This defines the resolving power of the scanning FP method

$$\frac{\lambda}{\delta\lambda} \sim \frac{\omega}{\delta\omega} \sim \frac{\nu}{\delta\nu} \sim N\mathcal{F}$$

see Fowles 4.2, Hecht 9.6.1.

Internal field

Due to the multireflection in a FP the internal field inside the FP can be much larger than the incident field! On resonance $\delta = 0$

$$E_T = \frac{t^2 E_0}{1-r^2} = t\,E_{int}$$

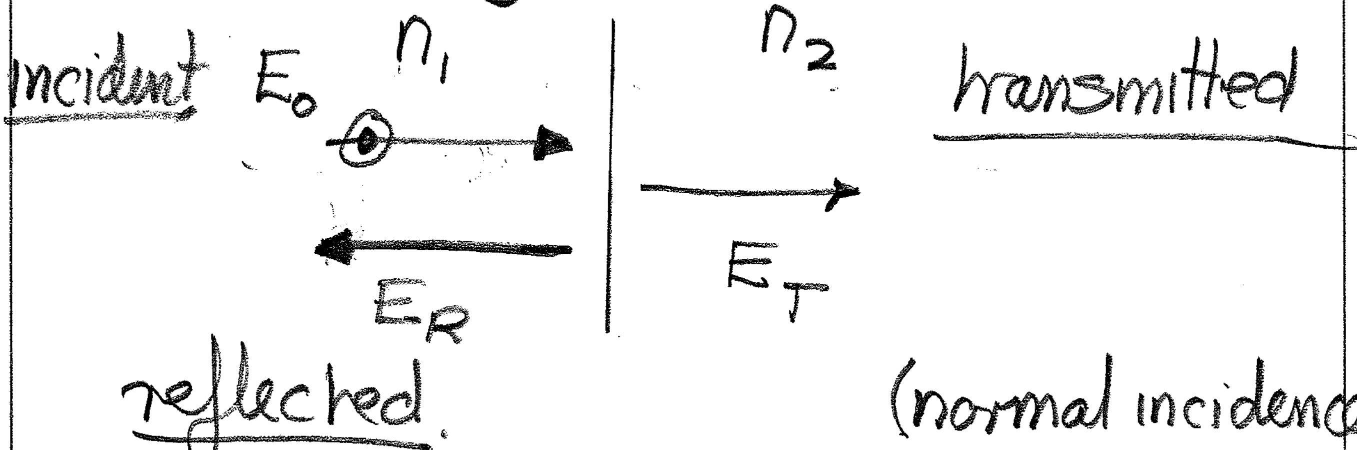
$$E_{int} = \frac{t\,E_0}{1-r^2} = \frac{\sqrt{T}\,E_0}{T}$$

$$= E_0/\sqrt{T} \gg E_0$$

for $R \sim 1$, $T \to 0$!

Multi-layer dielectric films. (Fowles 4.4)

The Fabry-Perot is such an amazing optical device it would be a shame if its interesting properties were confined to a device based on mirrors. FPs can also be made from stacks of dielectric films. Consider an interface with a normally incident beam



(normal incidence)

We know from Section 7 $(\theta = \phi = 0)$

$$r_s = r_p = \frac{(1-n)}{(1+n)}, \quad n = \frac{n_2}{n_1}$$

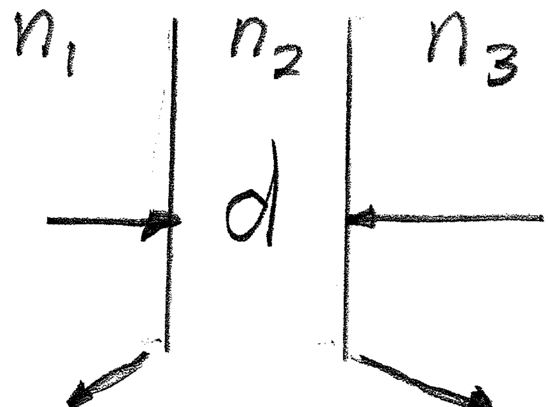
$$= \frac{(n_1 - n_2)}{(n_1 + n_2)} = r$$

Note that for $n_1 > n_2$, r is positive, whereas for $n_2 > n_1$, r is negative, giving a π phase-shift

$$r = |r|(-1) = |r|e^{i\pi}.$$

The key point is that an interface has a reflectivity (t is positive in all cases)

Now consider a <u>thin film</u>



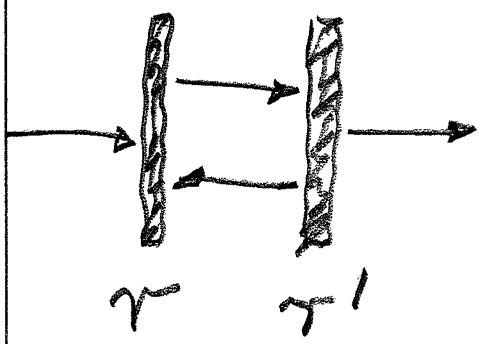
$\boxed{\text{two interfaces}}$

$$r = \frac{(n_1 - n_2)}{(n_1 + n_2)} \qquad r' = \frac{(n_2 - n_3)}{(n_2 + n_3)}$$

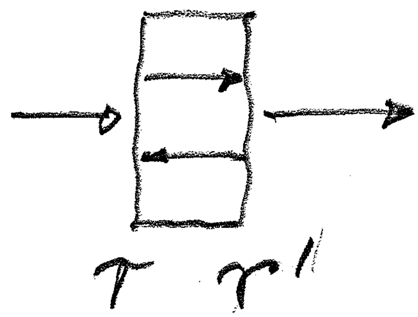
the film have thickness d and is sandwiched between to bulk media.

There are reflections at each dielectric interface so this thin film device can function as a FP using <u>Fresnel reflections</u>

<u>mirror FP</u>



<u>Thin film FP</u>



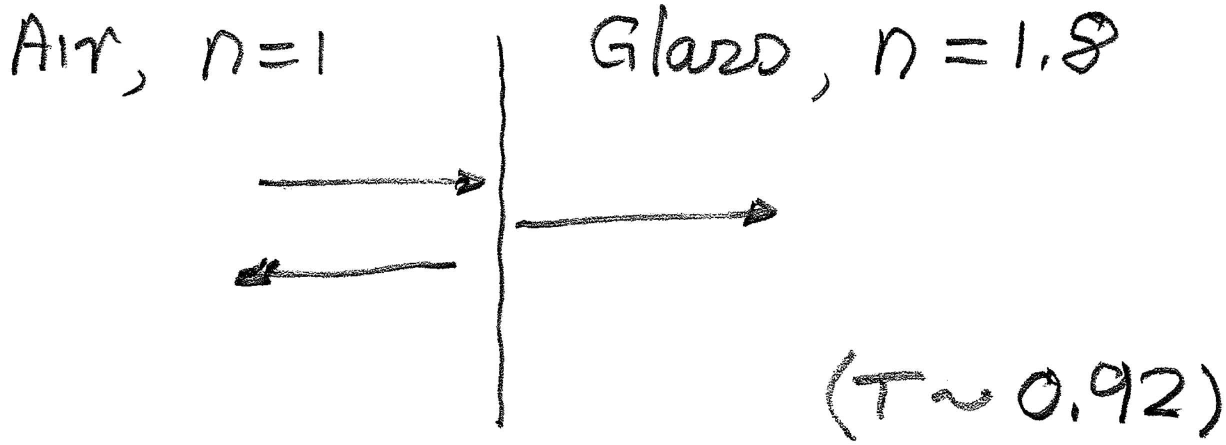
Recall that an FP can do two very interesting things

- Highly reflecting off-resonance
  → <u>thin film mirrors</u>
- Highly transmitting on-resonance.
  → <u>thin film anti-reflection (AR) coatings</u>

We now give a qualitative discussion of how these work,

AR coatings

How can we eliminate reflections at an interface, eg air & glass



The Fresnel reflection is $R \sim 0.08$.
Idea: insert a thin film or coating layer

$$1 < n_{coating} < n_{glass}.$$

of thickness $(\lambda/4)$

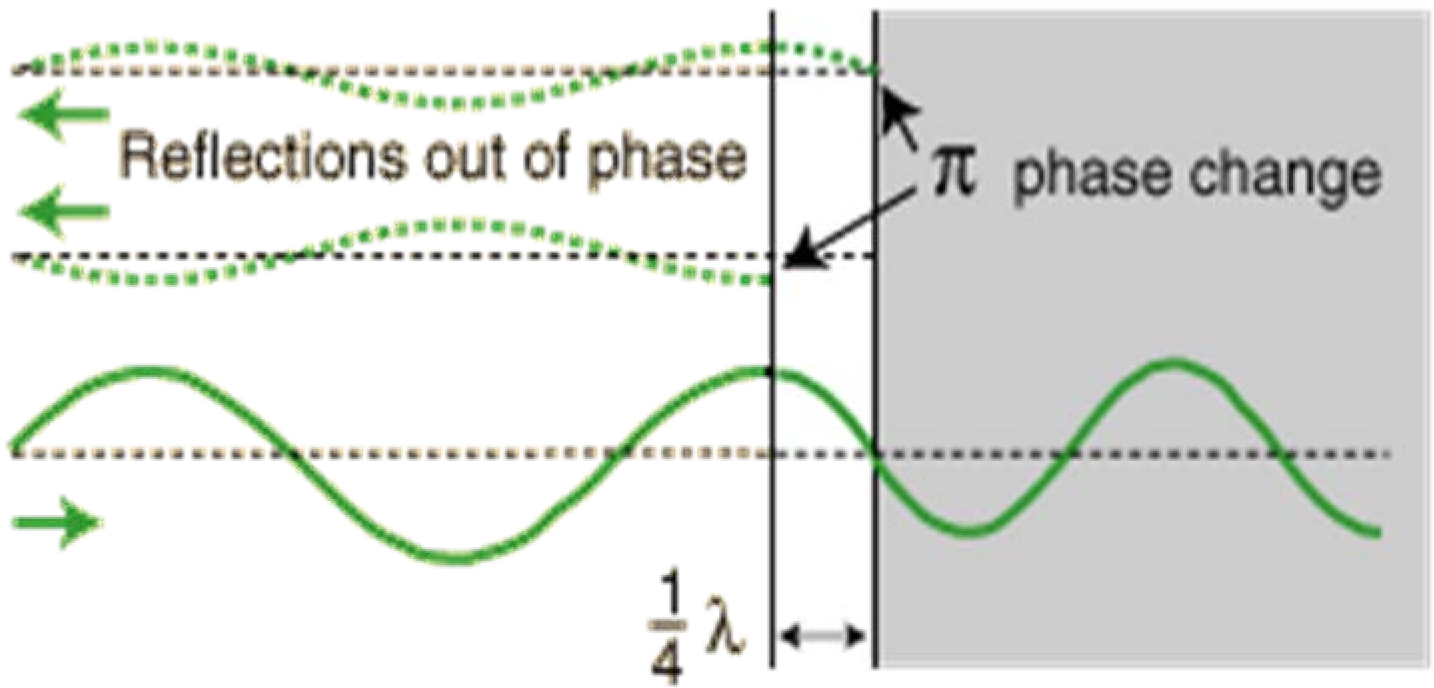
$$d = \left(\frac{\lambda}{4}\right) = \left(\frac{\lambda_0}{4 n_{coating}}\right)$$

this is a quarter-wave coating.
See figure for geometry.

Single Layer Antireflection Coating

- Concept: Quarter-wave coating

Anti-reflection coatings work by producing two reflections which interfere destructively with each other.



The propagation phase-shift is.

$$\varphi = k \cdot (2d) = \frac{2\pi}{\lambda} \cdot (2d) = \pi$$

So the reflected fields from the two interfaces tend to cancel

$\longrightarrow$ small reflected field!

The coated interface becomes highly transmitting - <u>AR coating</u>.

<u>Optimum coating</u> for

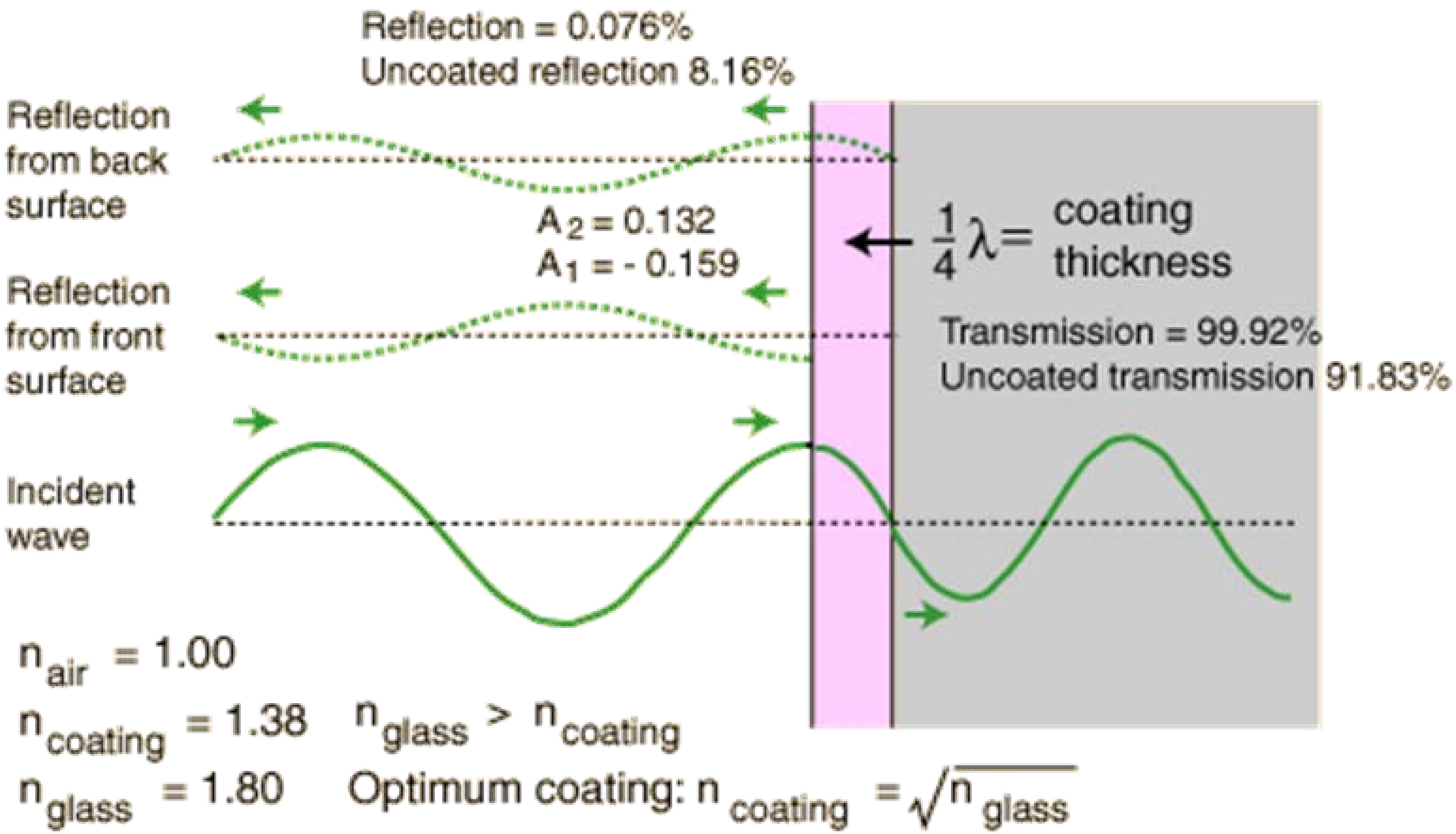
$$\underset{(R=0)}{n_{coating}} = \sqrt{n_{glass}}$$

then $T=1$ for a quarter-wave coating. (often not exactly possible)

See figure for example.

Single Layer Antireflection Coating

- Example: Quarter-wave coating



A limitation is that quarter-wave AR coating is designed for a given incident λ or ω due to the condition

$$d = \frac{\lambda}{4} = \frac{\lambda_0}{4n_{coating}}$$

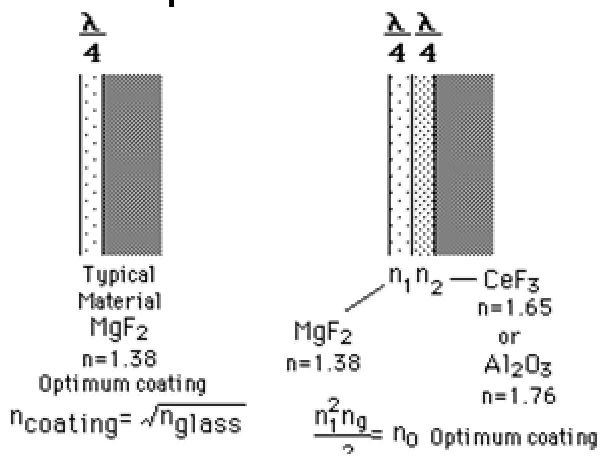
To adress this stacks of quarter-wave with alternating low and high refractive-index are employed - see figure.

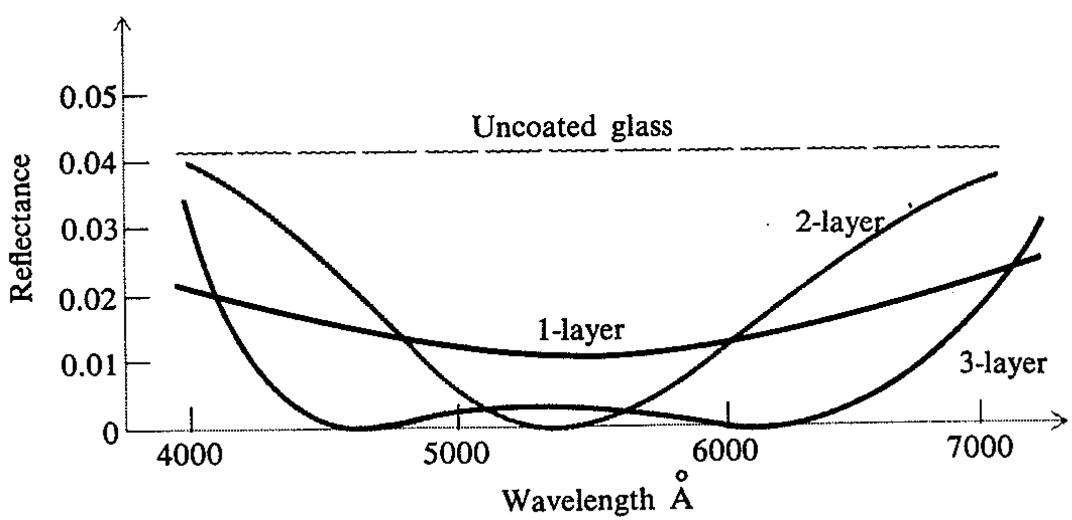
Multi-layer thin film AR coatings is a massive industry. For an optical system with several lenses (air glass interfaces) AR coating is essential to avoid large amounts of stray light due to unwanted reflections.

(Note - name thin film since their thickness is a fraction of a wavelength).

Multi-Layer Anti-Reflection Coatings

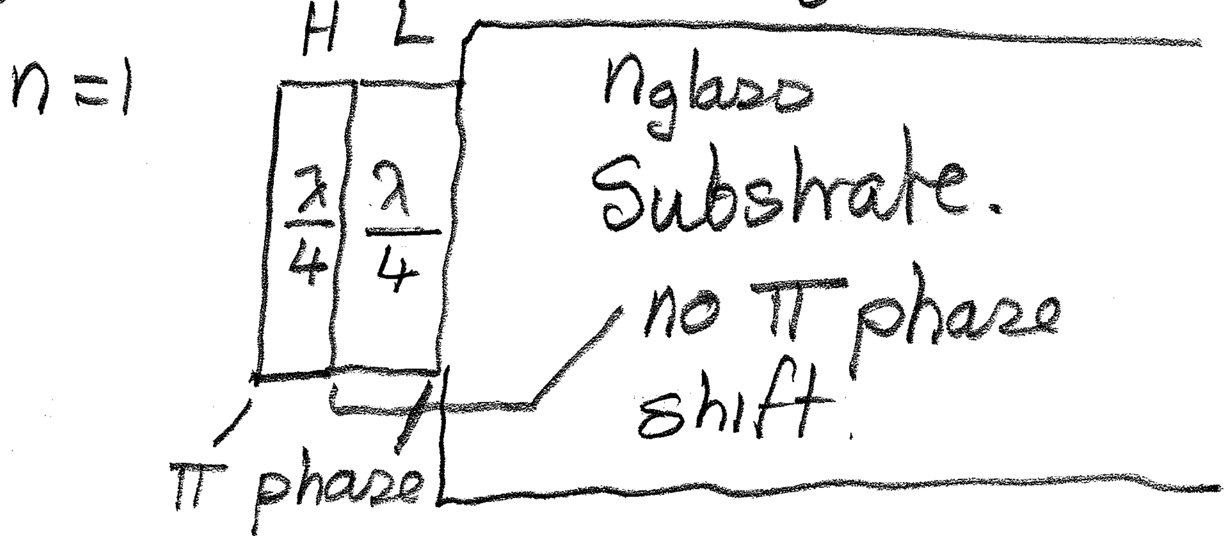
A single layer anti-reflection coating can be made non-reflective only at one wavelength, usually at the middle of the visible. Multiple layers are more effective over the entire visible spectrum.





High reflectance multi-layers

In this case one use two 1/4-wave layers as follows (now high-low).

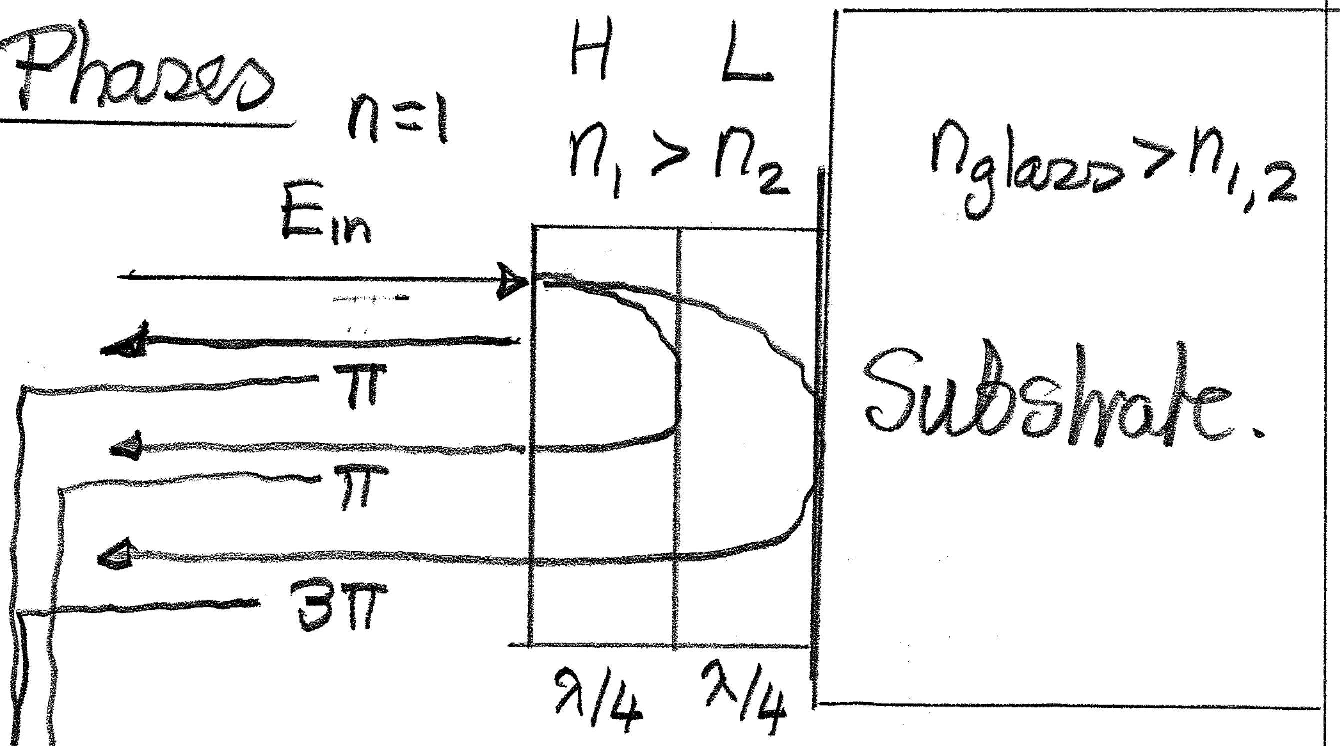


Now the propagation phase-shift is

$$\varphi = k \cdot 2(d_1 + d_2) = \pi + \pi = 2\pi$$

=> the back reflected waves add in phase - high reflectivity mirrors with ref HL $\lambda/4$ reflection coating - see phases on next page.

Phases



→ π (from reflection

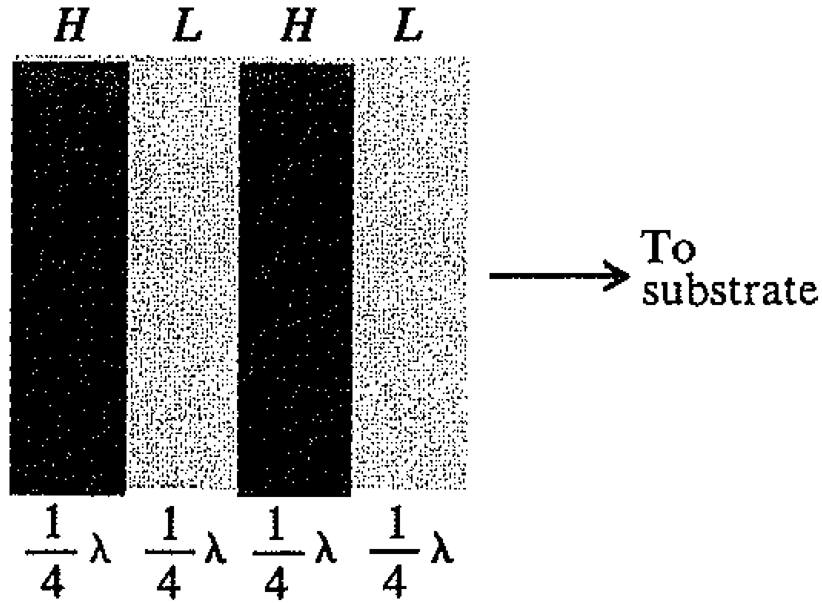
→ π (from propagation λ/4)

→ $3\pi = \pi$ (from back reflection)
$+ 2\pi$ (from propagation λ/2)

All back reflected waves have same phase — strong reflection!

High-reflectance multilayer films

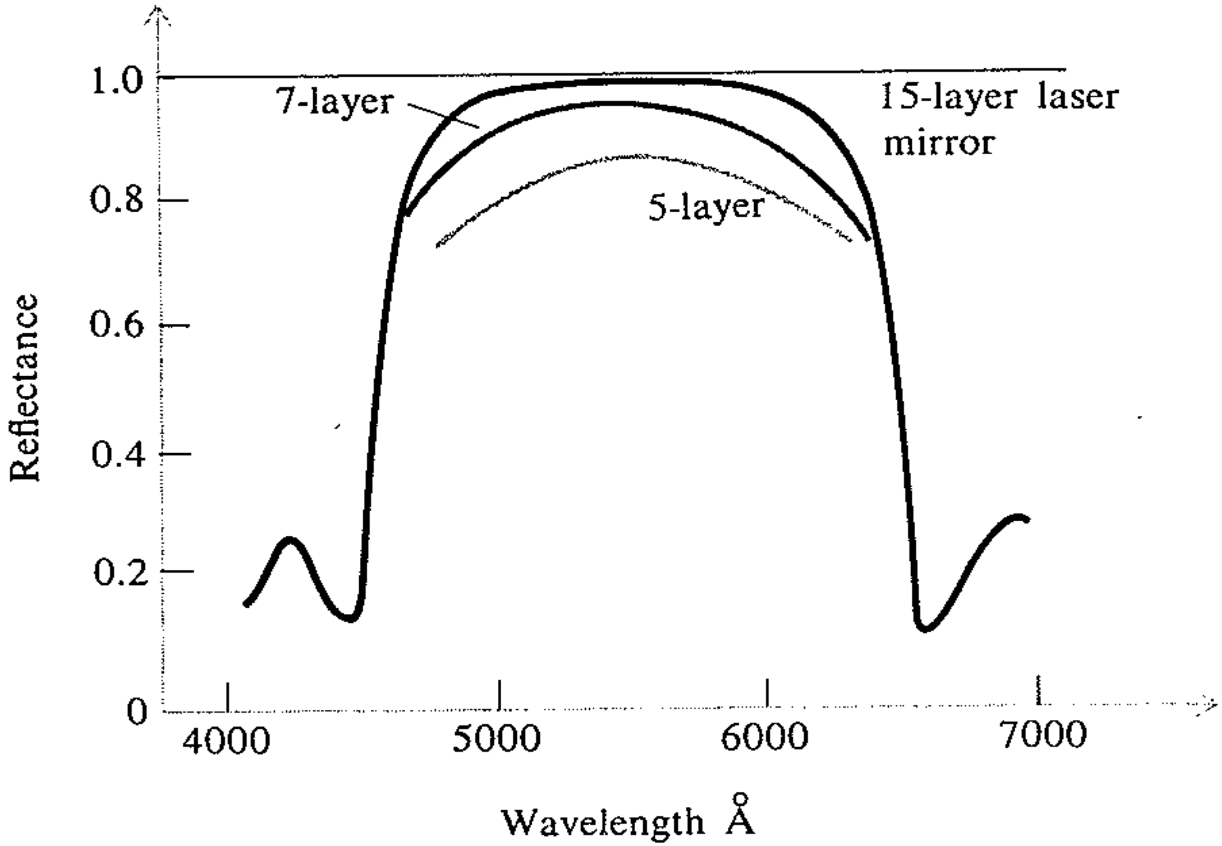
- Concept: Alternating quarter-wave layers



Multilayer stack for producing high reflectance. The stack consists of alternate quarter-wave layers of high and low index material. (Note: λ is the wavelength in the material.)

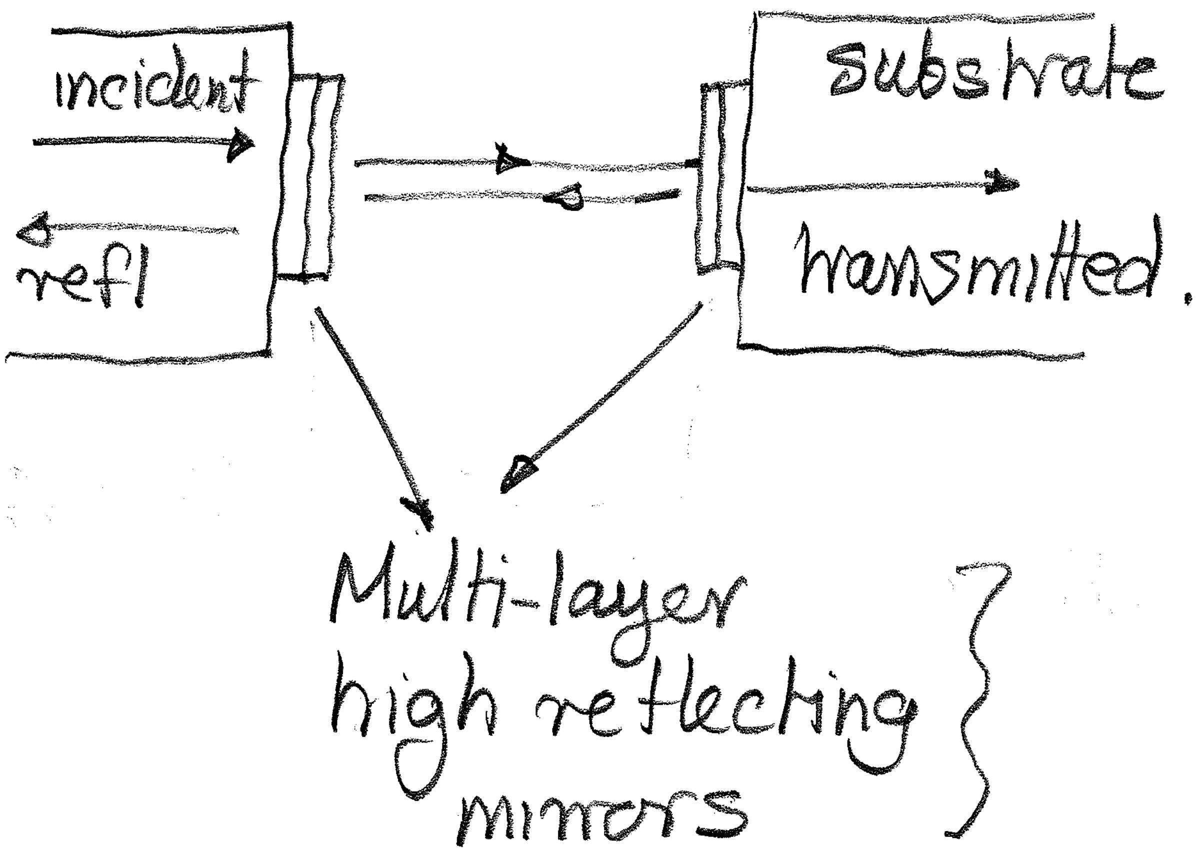
High-reflectance multilayer films

- Example: Alternating quarter-wave layers



Again one use multi-layer stacks to creating high reflectivity over a broad range of wavelengths — see figure.

Multi-layer films are actually often used to create the mirrors for Fabry-Perots



Can achieve high R & F