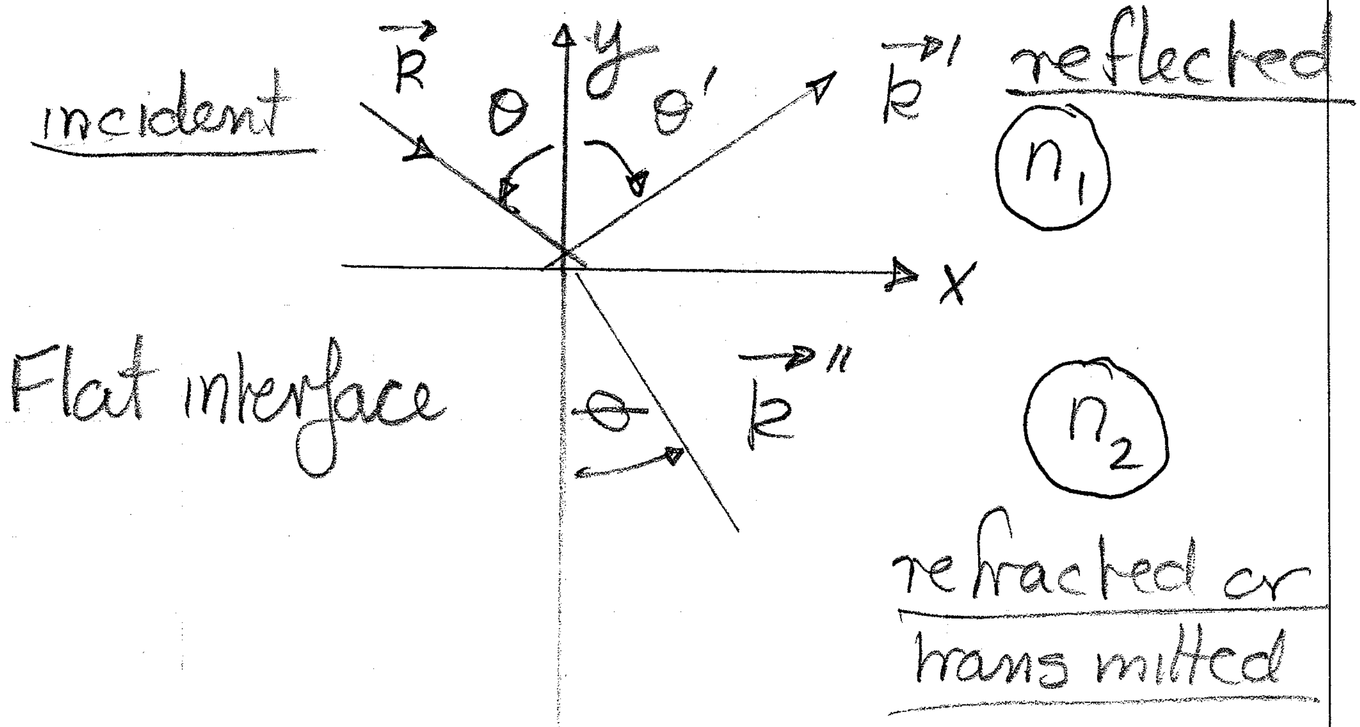
# 7. Dielectric Interface: General (Fowles 38–51)

Interface in x–z plane: see Fowles p. 40,
incident $\vec{k}$ in x–y plane.



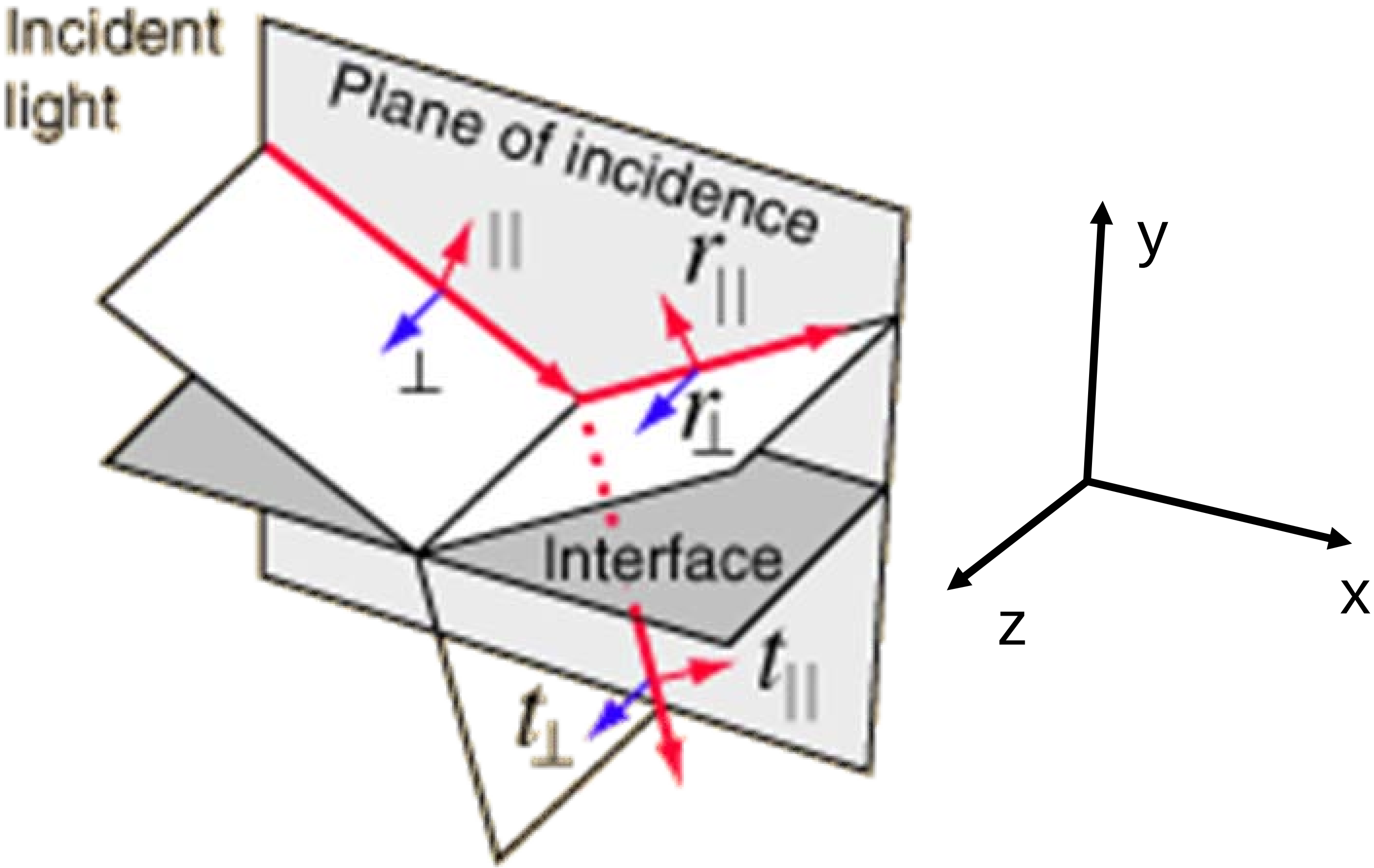

Incident, reflected & transmitted are all plane-waves eg.

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left[\vec{E}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + c.c\right]$$

$$\vec{E}'(\vec{r},t) = \frac{1}{2}\left[\vec{E}'(\vec{k}',\omega')e^{i(\vec{k}'\cdot\vec{r}-\omega' t)} + c.c\right]$$

Need to impose boundary conditions at interface to find relations between

# Dielectric Interface



incident, reflected & transmitted.

• Relations don't depend on time

$$\Rightarrow \omega = \omega' = \omega''$$

$$= \hbar\omega = \hbar\omega' = \hbar\omega'' \leftarrow \text{energy conserv.}$$

Can consider positive frequency comp. alone. (use Helmholtz solutions)

• Phase profiles agree on boundary

$$\Rightarrow \vec{k}\cdot\vec{r}_b = \vec{k}'\cdot\vec{r}_b = \vec{k}''\cdot\vec{r}_b = \text{const}.$$

$\Rightarrow$ $\vec{k}, \vec{k}', \vec{k}''$ are coplanar, in x-y plane in our geometry – plane of incidence

then eg. $(\vec{k}-\vec{k}')\cdot\vec{r}_b = 0$ etc.

Consider line on surface $(x,0,0)$

then ① ① ② ← medium

$$k_x X = k'_x X = k''_x X$$

eg. $\vec{k} = (k_x, k_y, k_z) = \dfrac{n_1\omega}{c}(\sin\theta, -\cos\theta, 0)$

$$\vec{k}' = \frac{n_1\omega}{c}(\sin\theta', \cos\theta', 0)$$

$$\vec{k}'' = \frac{n_2\omega}{c}(\sin\phi, -\cos\phi, 0)$$

Setting $k_x = k'_x$ gives

$$\frac{n_1\omega}{c}\sin\theta = \frac{n_1\omega}{c}\sin\theta'$$

or $\underline{\theta = \theta', \text{ law of reflection}}$

Setting $k_x = k''_x$

$$\frac{n_1\omega}{c}\sin\theta = \frac{n_2\omega}{c}\sin\phi$$

or

$$\boxed{n_1\sin\theta = n_2\sin\phi}, \quad \underline{\text{Snell's law of refraction}}$$

or

$$\boxed{\frac{\sin\theta}{\sin\phi} = n = \frac{n_2}{n_1}}$$

$n$ - relative index of refraction (Fowles)

## Critical angle

$$\frac{\sin\theta}{\sin\phi} = \frac{n_2}{n_1} = n$$

$\phi = \pi/2$, no transmitted field for $\theta > \theta_{cr}$ with

$$\sin\theta_{cr} = n_2/n_1 = n$$

$$\boxed{\theta_{cr} = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}(n)}$$

Happens if $n_1 > n_2 \rightarrow$ internal reflection (TIR for $\theta > \theta_{cr}$).

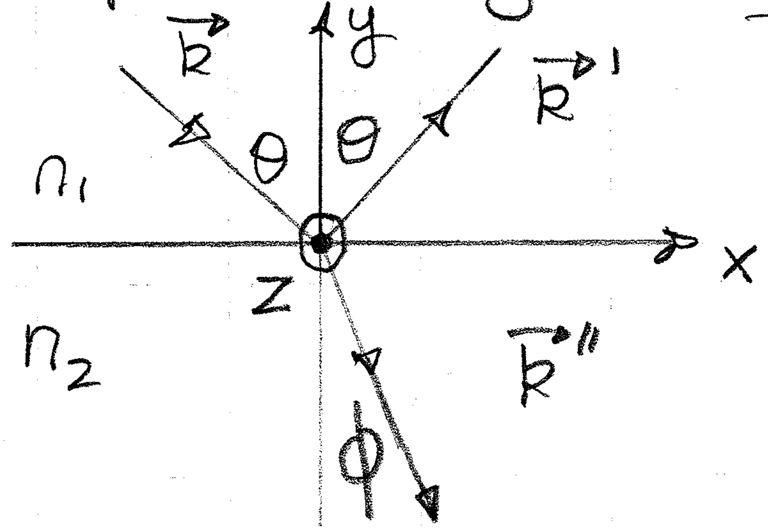
External reflection $n_1 < n_2$

$n > 1$

$$\boxed{n = \frac{n_2}{n_1}, \text{ relative refractive-index.}}$$

## Reflected & refracted fields

Consider a planar interface (Fowles 38-51)




x-z plane of interface

x-y plane of incidence containing $\vec{k}, \vec{k}', \vec{k}''$

Need to solve Helmholtz equations

$$\left[\nabla^2 + n_1^2\omega^2/c^2\right]\vec{E}(\vec{r},\omega) = 0 \ , \quad y > 0$$

$$\left[\nabla^2 + n_2^2\omega^2/c^2\right]\vec{E}(\vec{r},\omega) = 0 \ , \quad y < 0$$

In each region use plane-wave solns.

$$\vec{E}(\vec{r},\omega) = \underbrace{\vec{E}(\vec{k},\omega)e^{i\vec{k}\cdot\vec{r}}}_{\text{incident}} + \underbrace{\vec{E}(\vec{k}',\omega)e^{i\vec{k}'\cdot\vec{r}}}_{\text{reflected}} \qquad y > 0$$

$$\vec{E}(\vec{r},\omega) = \vec{E}(\vec{k}'',\omega)\, e^{i\vec{k}''\cdot\vec{r}} \quad , \; y \leq 0$$

transmitted.

Can get corresponding magnetic fields from Faraday's law in $(\vec{k},\omega)$ space

$$i\vec{K} \times \vec{E}(\vec{K},\omega) = i\omega\mu_0 \vec{H}(\vec{K},\omega)$$

$\vec{K} = \vec{k}, \vec{k}', \vec{k}''$

$$\vec{H}(\vec{k},\omega) = \frac{1}{\omega\mu_0} \vec{k} \times \vec{E}(\vec{k},\omega)$$

unit vector

$$= n_1 c \epsilon_0 \hat{k} \times \vec{E}(\vec{k},\omega)$$

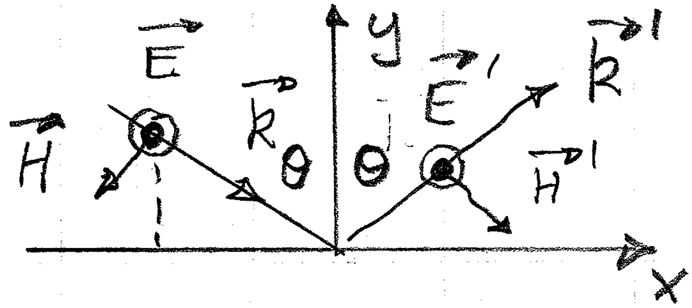
$$\vec{H}(\vec{k}',\omega) = n_1 c \epsilon_0 \hat{k}' \times \vec{E}(\vec{k}',\omega)$$

$$\vec{H}(\vec{k}'',\omega) = n_2 c \epsilon_0 \hat{k}'' \times \vec{E}(\vec{k}'',\omega)$$

Need to impose boundary conditions to find relation between $\vec{E}(\vec{k},\omega)$ $\vec{E}(\vec{k}',\omega)$ etc.

## s- and p-polarized incident waves

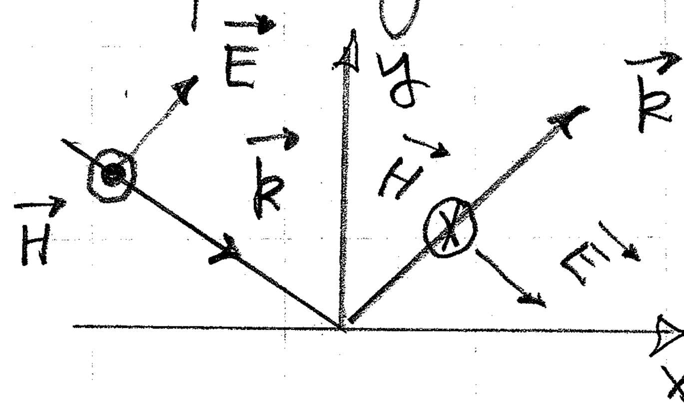
s-polarized incident wave, $\vec{E}(\vec{k},\omega)$ perpendicular (senkrecht) to plane of incidence (x-y)




p. 42 Fowles

also called Transverse Electric (TE) polarization => $\vec{E} \perp$ to plane of incidence

p-polarized incident wave, $\vec{E}(\vec{k},\omega)$ parallel to plane of incidence (x, y)

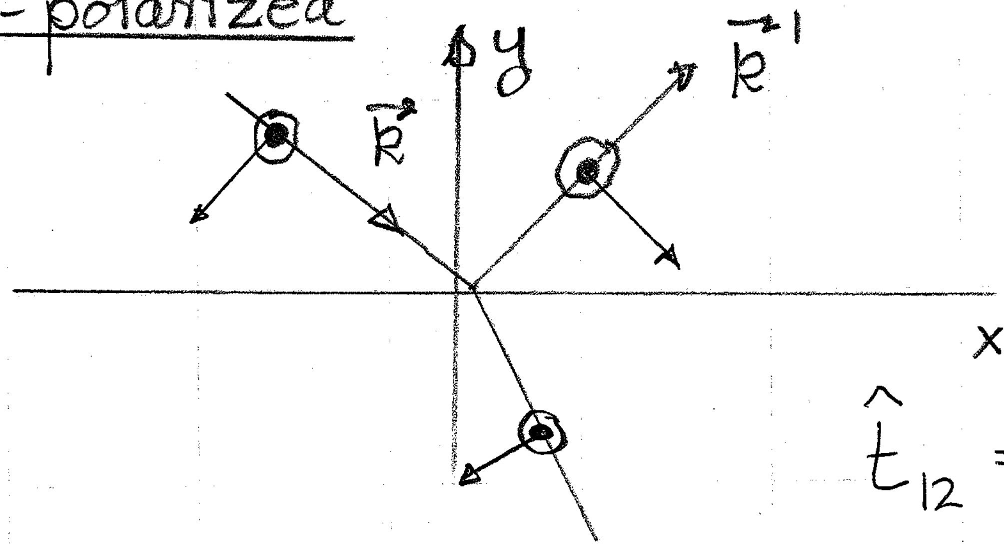



also called Transverse magnetic (TM) polzn => $\vec{H} \perp$ to plane of incidence.

Any input plane-wave of wavecter $\vec{k}$ can be decomposed into s- and p-polarized components

Reflection and transmission coefficients

- <u>s-polarized</u>




$\hat{t}_{12} = \hat{z}$ or $\hat{x}$

$\vec{E}$ is in $\hat{z}$ direction and is tangential to the interface in the $(x-z)$ plane

$$\vec{E}(\vec{r},\omega) = \hat{z} E e^{i\vec{k}\cdot\vec{r}} + \hat{z} E' e^{i\vec{k}'\cdot\vec{r}} \qquad y>0$$

$$\vec{E}(\vec{r},\omega) = \hat{z} E'' e^{i\vec{k}''\cdot\vec{r}} , \qquad y<0$$

$E = \hat{z}\cdot\vec{E}(\vec{k},\omega)$ - incident field amplitude

$E' = \hat{z}\cdot\vec{E}(\vec{k}',\omega)$ - reflected " "

$E'' = \hat{z} \cdot \vec{E}(\vec{k}'', \omega)$ – transmitted field ampl.

Here we choose $\hat{t}_{12} = \hat{z}$, so continuity of $\hat{t}_{12} \cdot \vec{E}$ at the interface gives

$$E e^{i\vec{k} \cdot \vec{r}_b} + E' e^{i\vec{k}' \cdot \vec{r}_b} = E'' e^{i\vec{k}'' \cdot \vec{r}_b}$$

$$\Rightarrow \boxed{E + E' = E''}$$

Now look at $\vec{H}$ for s-polarized

$$\vec{H}(\vec{k}, \omega) = n_1 c \epsilon_0 \hat{k} \times \vec{E}(\vec{k}, \omega)$$

$$\equiv n_1 c \epsilon_0 E \, \hat{k} \times \hat{z}$$

$$\vec{H}(\vec{k}', \omega) = n_1 c \epsilon_0 E' \hat{k}' \times \hat{z}$$

$$\vec{H}(\vec{k}'', \omega) = n_2 c \epsilon_0 E'' \hat{k}'' \times \hat{z}$$

Continuity of $\underbrace{\hat{t}_{12} \cdot \vec{H}}_{\text{tangential component of } \vec{H}} \equiv \hat{x} \cdot \vec{H}$, $\hat{t}_{12} \equiv \hat{x}$

across the boundary, eg

$$\hat{x} \cdot \vec{H}(\vec{k}, \omega) = n_1 c \epsilon_0 E \, \hat{x} \cdot \hat{k} \times \hat{z}$$

$$= n_1 c \epsilon_0 E \, \hat{y} \cdot \hat{k} = n_1 c \epsilon_0 E \left( \frac{k_y}{|\vec{k}|} \right)$$

$$= - n_1 c \epsilon_0 E \cos\theta$$

$$\hat{x} \cdot \vec{H}(\vec{k}', \omega) = n_1 c \epsilon_0 E' \cos\theta$$

$$\hat{x} \cdot \vec{H}(\vec{k}'', \omega) = - n_2 c \epsilon_0 E'' \cos\phi$$

Continuity of tangential comp. of $\vec{H}$.

$$\boxed{\begin{gathered} -n_1 E \cos\theta + n_1 E' \cos\theta = -n_2 E'' \cos\phi \\ E + E' = E'' \end{gathered}}$$

Get reflected field, $E'' = E + E'$

$$-n_1 E\cos\theta + n_1 E'\cos\theta = -n_2(E+E')\cos\phi$$

$$\Rightarrow \boxed{r_s = \left(\frac{E'}{E}\right) = \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi}}$$

$n = n_2/n_1$ as before

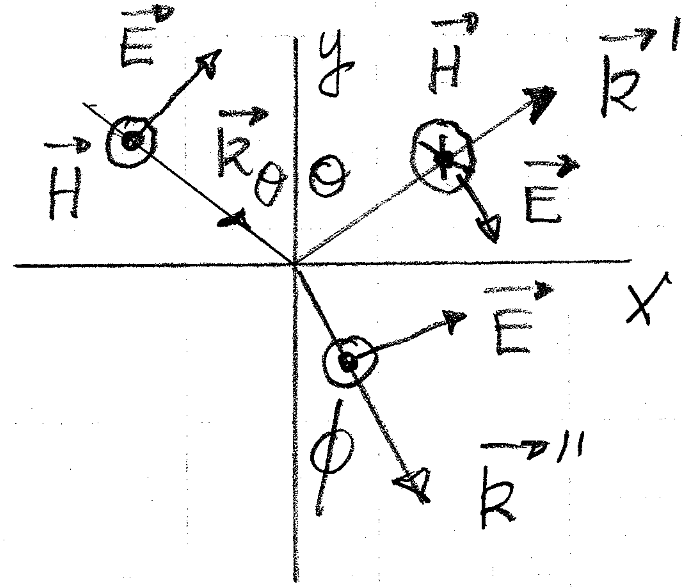
$r_s$ - refl coeff for s- or TE polarized.

In a similar manner $(E' = E'' - E)$.

$$\boxed{t_s = \frac{2\cos\theta}{\cos\theta + n\cos\phi} = \left(\frac{E''}{E}\right)}$$

$t_s$ - trans. coeff for s- or TE polarization.

- <u>p-polarized</u>: Follow same procedure



$$\vec{H}(\vec{r},\omega) = \hat{z} H e^{i\vec{k}\cdot\vec{r}} - \hat{z} H' e^{i\vec{k}'\cdot\vec{r}} \qquad y>0$$

$$\vec{H}(\vec{r},\omega) = \hat{z} H'' e^{i\vec{k}''\cdot\vec{r}} \qquad y<0.$$

$H \equiv \vec{H}(\vec{k},\omega)\cdot\hat{z}$ etc

Continuity of $\vec{H}$ on the interface gives

$$H - H' = H''$$

Now $H = \hat{z}\cdot\vec{H}(\vec{k},\omega) = n_1 c \epsilon_0 \hat{z}\cdot(\hat{k}\times\vec{E})$

$= n_1 c \epsilon_0 (\hat{z}\times\hat{k})\cdot\vec{E}$

$\hat{z}\times\hat{k}$ – unit vector in $\vec{E}$ direction

$$H = n_1 c \epsilon_0 E$$ — amplitude of E field.

similiary $H' = n_1 c \epsilon_0 E'$, $H'' = n_2 c \epsilon_0 E''$

$$\longrightarrow \boxed{n_1 E - n_1 E' = n_2 E''}$$

results from continuity of $\vec{H}$

Next continuity of tangential component of $\vec{E}$, $\hat{t}_{12} = \hat{x}$

$$\hat{x} \cdot \left( \vec{E}(\vec{r},\omega) + \vec{E}'(\vec{r}',\omega) \right) = \hat{x} \cdot \vec{E}(\vec{r}'',\omega)$$

$$\rightarrow \boxed{\begin{array}{c} E\cos\theta + E'\cos\theta = E''\cos\phi \\ n_1 E - n_1 E' = n_2 E'' \end{array}}$$

This yields

$$\boxed{r_p = \frac{\cos\phi - n\cos\theta}{\cos\phi + n\cos\theta}}$$

→ refl. coeff for p-polarized or TM.

Leave calculation of $t_p$ to you!

$$\boxed{t_p = \frac{2\cos\theta}{\cos\phi + n\cos\theta}}$$

## Normal incidence

For normal incidence $\theta = \phi = 0$

$$r_s = \frac{1-n}{1+n} \quad , \quad t_s = \frac{2}{1+n}$$

$\pi$ phase-shift on reflection for $n > 1 \rightarrow$ external reflection, $n = n_2/n_1$

Also $\boxed{r_p = r_s \ , \ t_p = t_s}$. The intensity reflectivity & transmittivity

$$R = \frac{n_1 E'^2}{n_1 E^2} = \left(\frac{n-1}{n+1}\right)^2 = r^2$$

$$T = \frac{n_2 E''^2}{n_1 E^2} = \frac{4n}{(1+n)^2} = nt^2$$

Have conservation of energy

$$T + R = 1$$

## Fresnel's equations

Classic form: use Snell's law $n = \frac{\sin\theta}{\sin\phi}$ to eliminate $n$ from eqns. for $r_s, t_s, r_p, t_p$

$$r_s = -\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)}, \quad t_s = \frac{2\cos\theta\sin\phi}{\sin(\theta+\phi)}$$

$$r_p = -\frac{\tan(\theta-\phi)}{\tan(\theta+\phi)}, \quad t_p = \frac{2\cos\theta\sin\phi}{\sin(\theta+\phi)\cos(\theta-\phi)}$$

used trig identities. (p.44 Fowles)

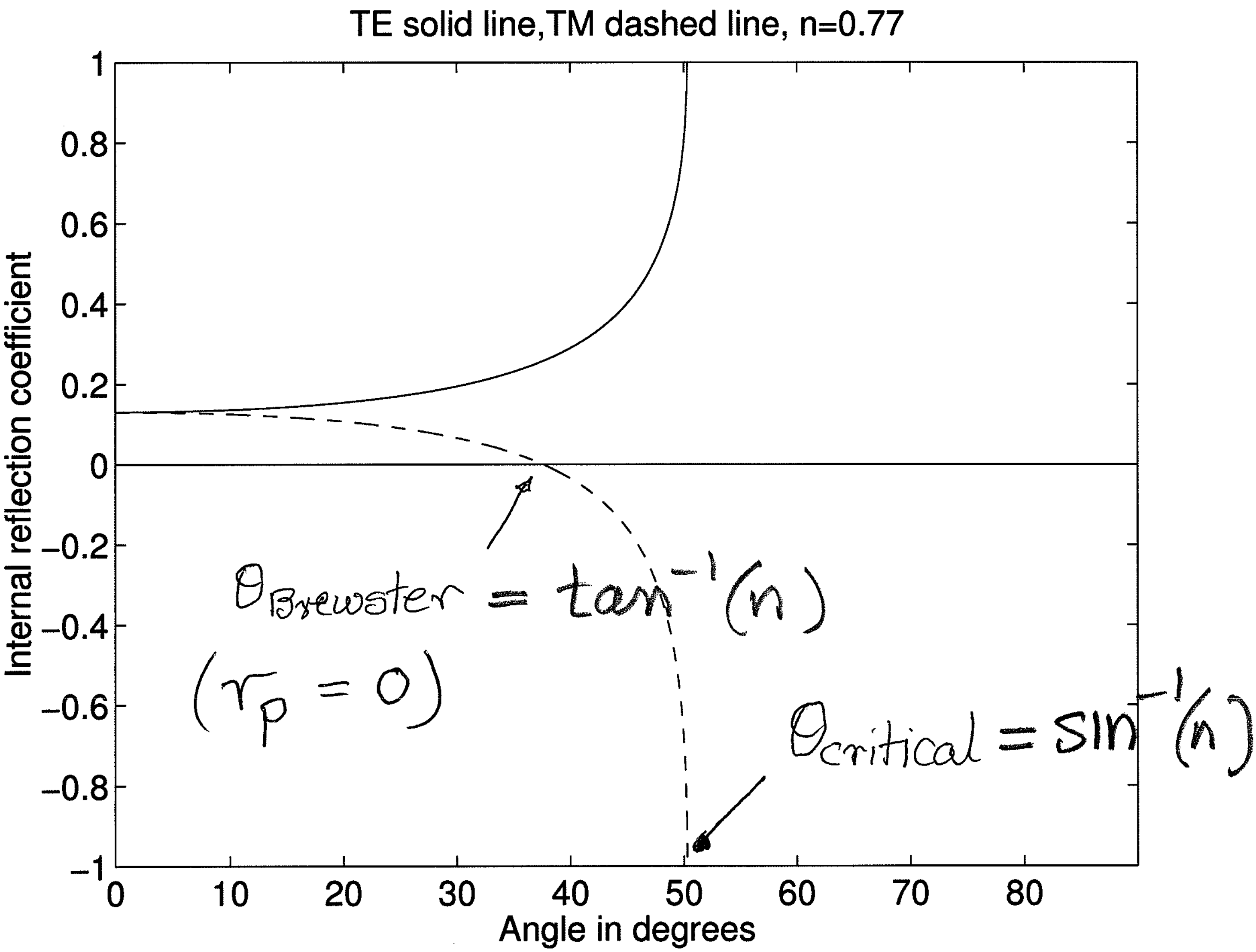
Useful form: use $n = \sin\theta/\sin\phi$ to eliminate $\phi$, eg.

$$r_s = \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi}$$

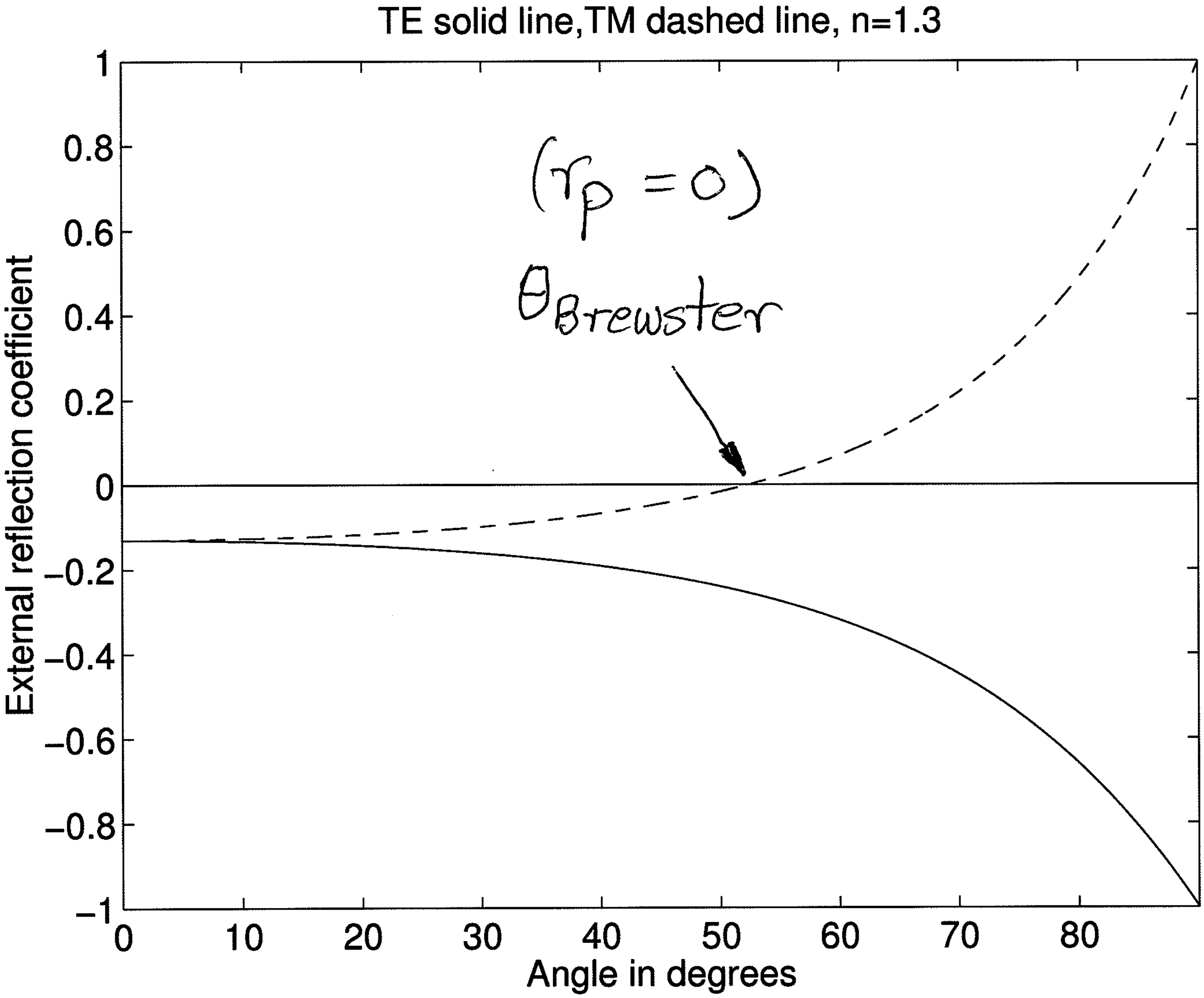
$\sin\phi = \sin\theta/n, \quad \cos\phi = \sqrt{1-\sin^2\theta/n^2}$

$n\cos\phi = \sqrt{n^2-\sin^2\theta}$

$$\Rightarrow r_s = \frac{\cos\theta - \sqrt{n^2-\sin^2\theta}}{\cos\theta + \sqrt{n^2-\sin^2\theta}} = r_s(\theta)$$




Brewster angle also called the polarizing angle.




$$\theta_{Brewster} = \tan^{-1}(n)$$

$r_s(\theta)$ as function of incident angle.
Similarly

$$r_p(\theta) = \frac{-n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}$$

+ similar for $t_s, t_p$.

<u>Intensity reflectivity or reflectance.</u>

Concentrate on reflection

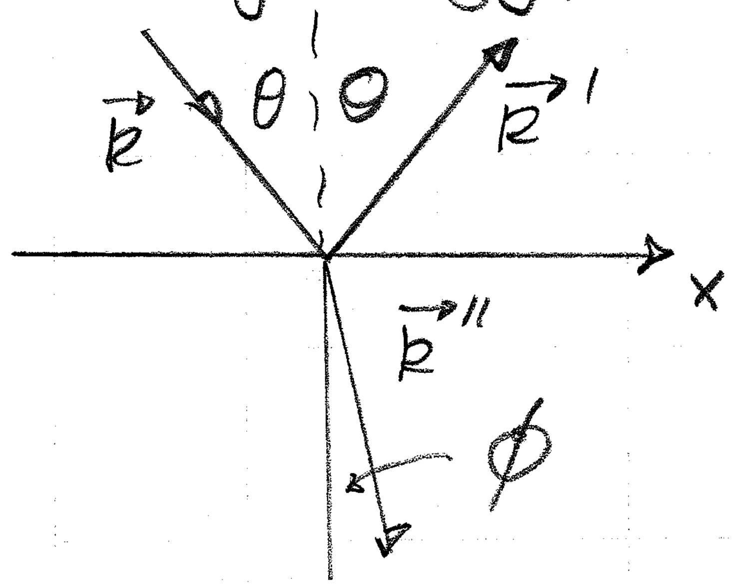
$$R = \frac{\frac{1}{2}\epsilon_0 n_1 c|E'|^2}{\frac{1}{2}\epsilon_0 n_1 c|E|^2} = \left|\frac{E'}{E}\right|^2$$

$$R_s = |r_s|^2, \quad R_p = |r_p|^2$$

For an unpolarized incident field with random superposition of s-p-polarization

$$\bar{R}(\theta) = \left(R_s(\theta) + R_p(\theta)\right)/2$$

## Conservation of energy




We have calculated $r = (E'/E)$, $t = (E''/E)$.

$\vec{S} = \frac{1}{2}\epsilon_0 c n_1 E^2 \hat{k}$, $\vec{S}' = \frac{1}{2}\epsilon_0 c n_1 E'^2 \hat{k}'$

$$\vec{S}'' = \frac{1}{2}\epsilon_0 c n_2 E''^2 \hat{k}''$$

Look at projection onto $\hat{y}$

$$S_y = \frac{1}{2}\epsilon_0 c n_1 E^2 (-\cos\theta) \qquad = \hat{k}\cdot\hat{y} = \frac{k_y}{|\vec{k}|}$$

$$S'_y = \frac{1}{2}\epsilon_0 c n_1 (E'^2) \cos\theta$$

$$S''_y = \frac{1}{2}\epsilon_0 c n_2 (E'')^2 (-\cos\phi)$$

$$R = \left| \frac{S_y'}{S_y} \right| = |r|^2$$

$$T = \left| \frac{S_y''}{S_y} \right| = \frac{n_2}{n_1} \left| \frac{E''}{E} \right|^2 \frac{\cos\phi}{\cos\theta}$$

$$= n t^2 (\cos\phi / \cos\theta)$$

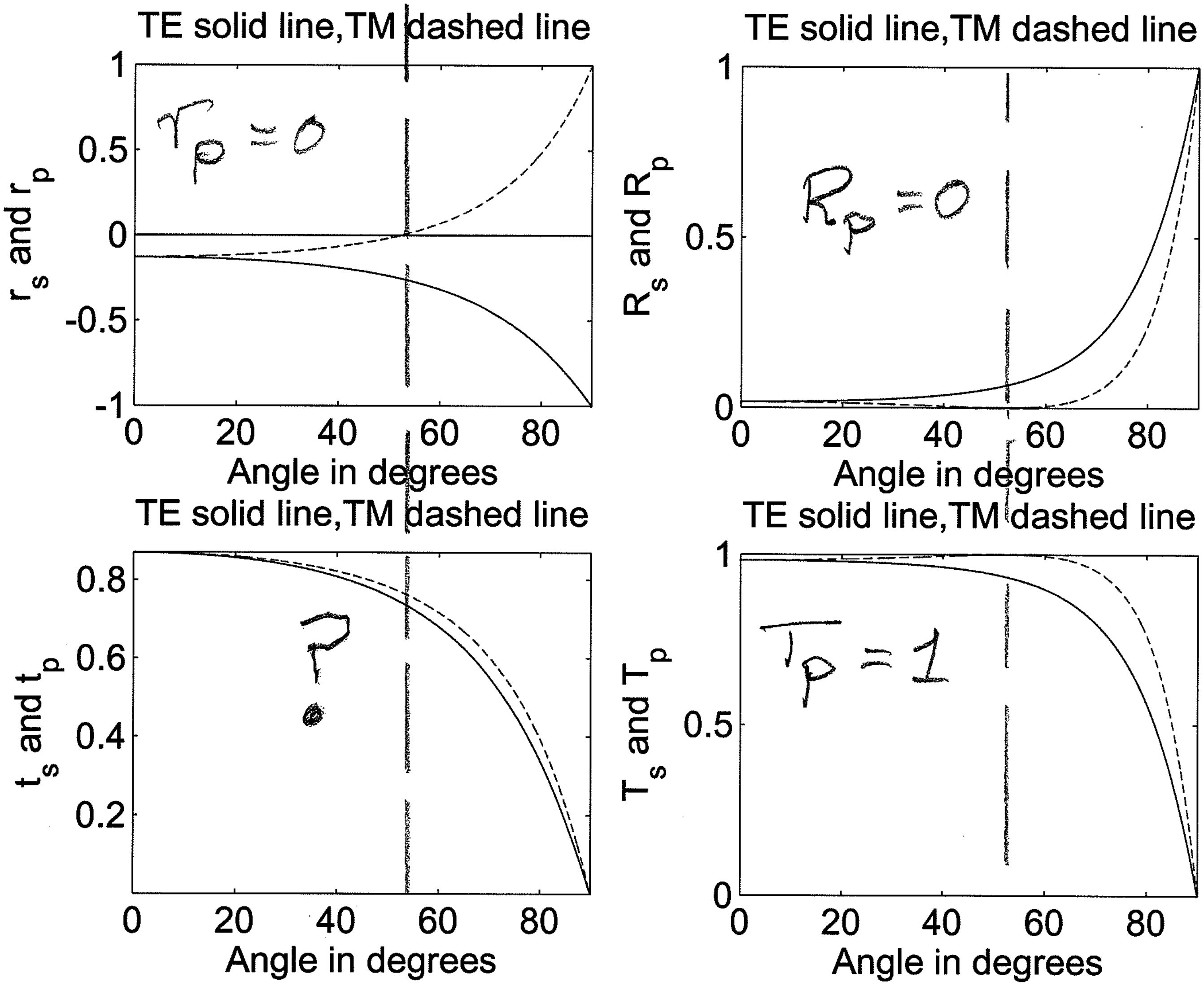
(due to change of direction & projection of cross-sectional area).

Check

$$R + T = 1$$

$$n = 1.3$$

$$R = |r|^2 , \quad T = |t|^2 n (\cos\phi / \cos\theta)$$



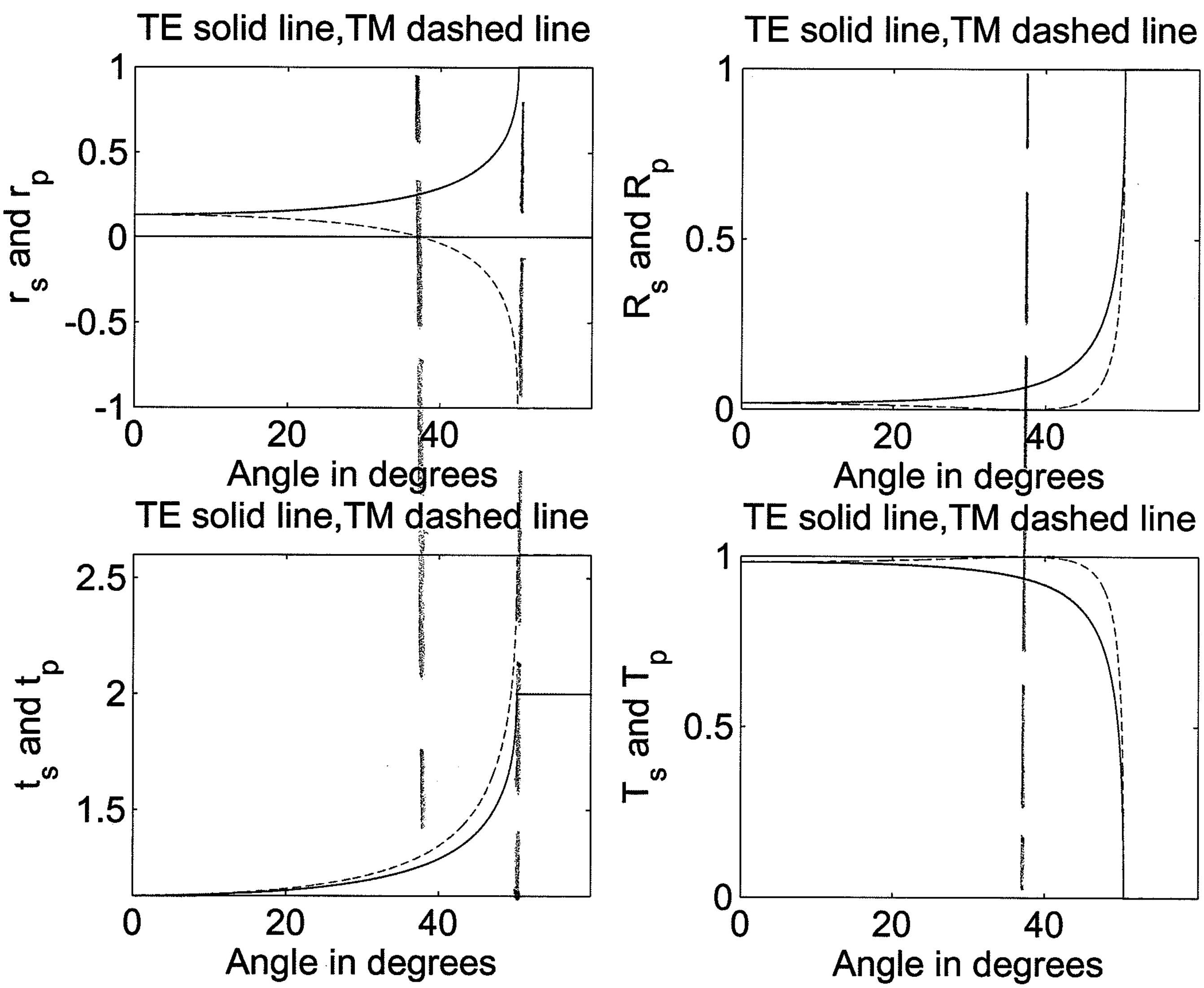

$$r = (1-n)/(1+n) = -0.13$$

$$t = 2/(1+n) = 0.87$$

$$\theta_{Br} \approx 52.5^\circ$$

$n = 1/1.3$

$R = |r|^2, \quad T = |t|^2 n(\cos\phi / \cos\theta)$



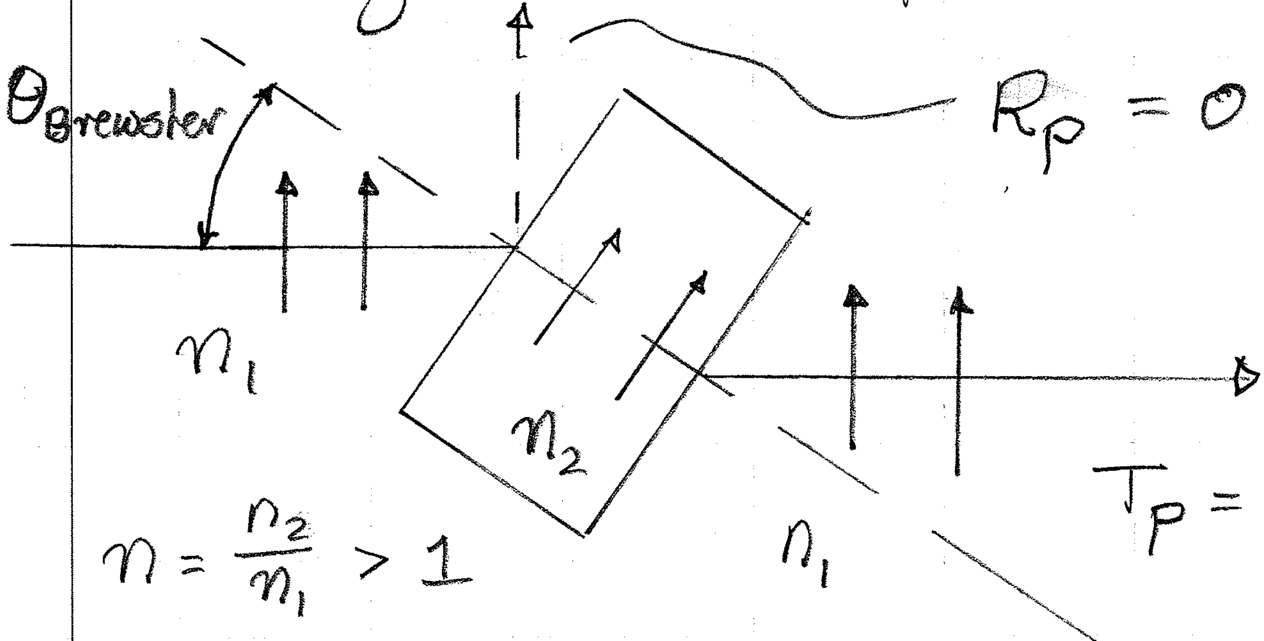

$r = (1-n)/(1+n) = 0.13$

$t = 2/(1+n) = 1.13$

$\theta_{Br} = 37.6^\circ, \; \theta_{cr} \approx 50^\circ$

## Brewster windows

Selectively transmit TM polarization.




$n = \frac{n_2}{n_1} > 1$

perfect window for TM.
imperfect " " " TE

TE suffers reflection losses —
used eg. polarization selection in lasers.

$$r_p(\theta_{Brewster}) = 0 \Rightarrow \theta_{Brewster} = \tan^{-1}(n)$$

$$\Rightarrow -n^2\cos\theta + \sqrt{n^2 - \sin^2\theta} = 0$$

$$n^4\cos^2\theta = n^2(\cos^2\theta + \sin^2\theta) - \sin^2\theta$$

$$(n^4 - n^2) = (n^2 - 1)\tan^2\theta, \quad \tan^2\theta = n^2$$

# Total internal reflection (TIR)

Internal reflection $n = n_2/n_1 < 1$,

eg.

$$r_s = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}},$$

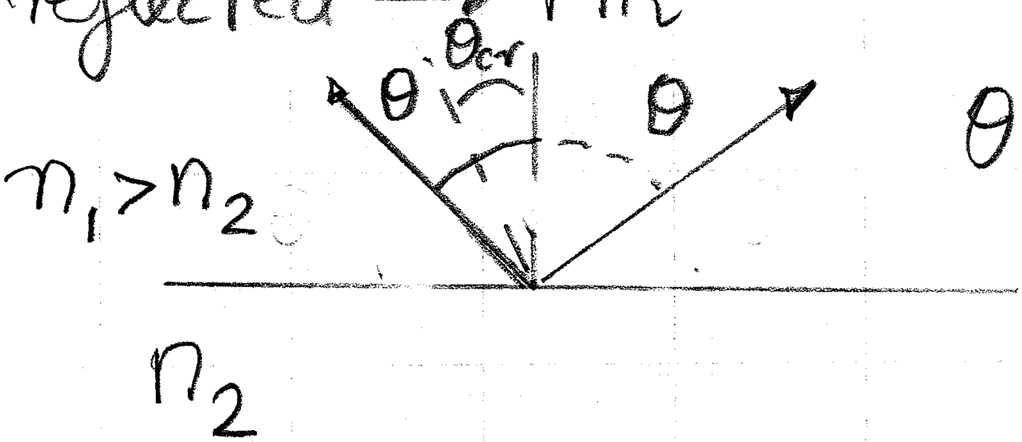
$0 < \theta < \pi/2$. If $\theta > \theta_{cr}$, $\sin\theta > n$

$$\sin\theta_{cr} = n, \quad \theta_{cr} = \sin^{-1}(n)$$

$\theta > \theta_{cr}$

$$r_s = \frac{\cos\theta - i\sqrt{\sin^2\theta - n^2}}{\cos\theta + i\sqrt{\sin^2\theta - n^2}}$$

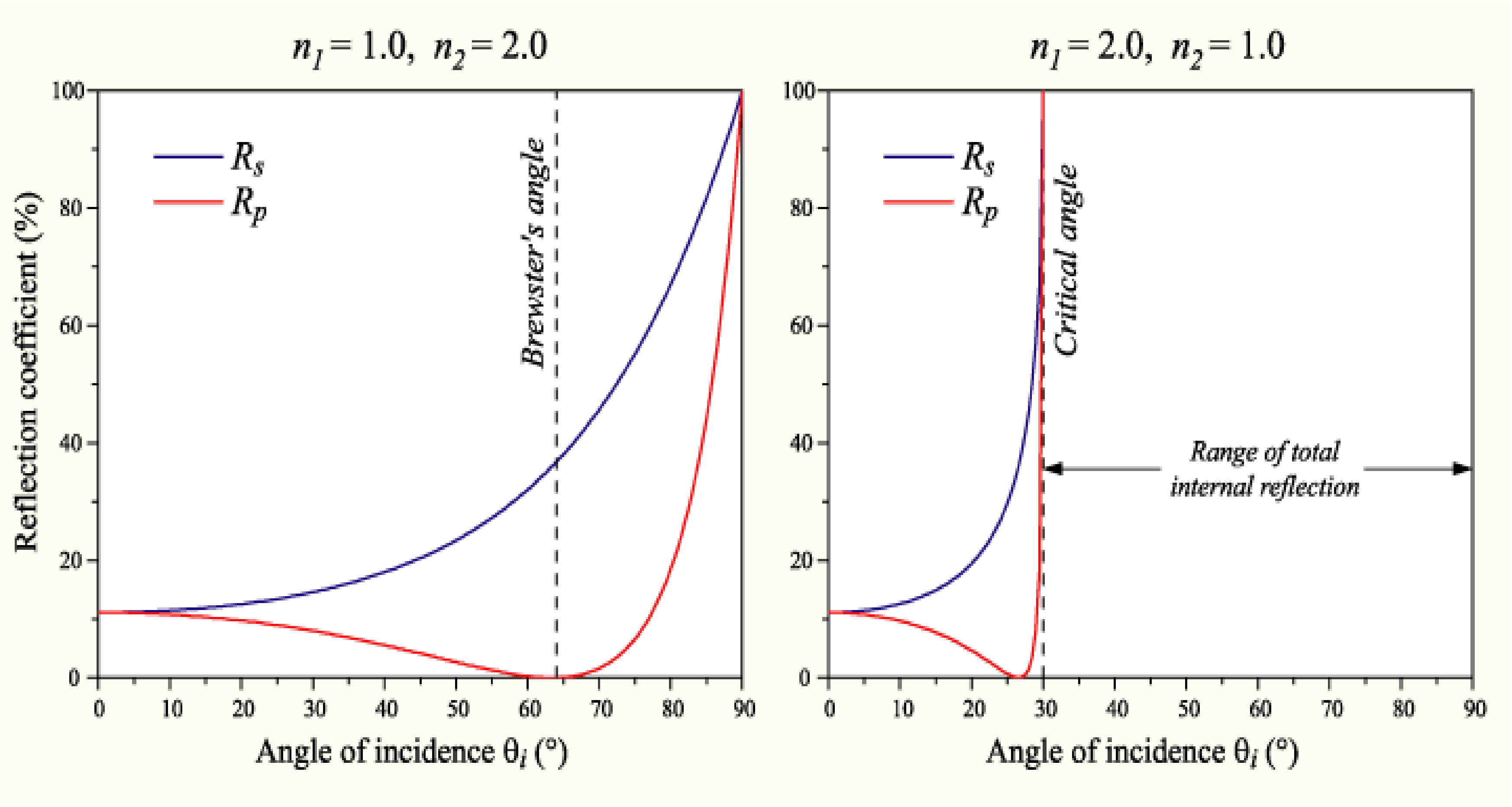
$|r_s|^2 = R_s = 1$, light is totally reflected $\longrightarrow$ TIR




$\theta > \theta_{cr}$

# Reflectance plots

External reflection | Internal reflection



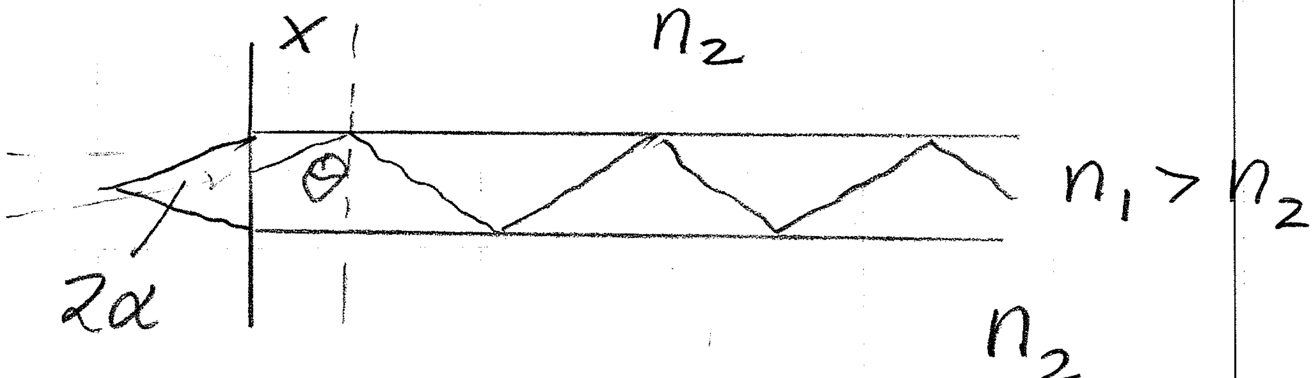
Same applies for p-polarized or TM

$$r_p = \frac{-n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}}$$

$$R_p = |r_p|^2 = 1 \,, \quad \theta > \theta_{cr}$$

## TIR and optical waveguides

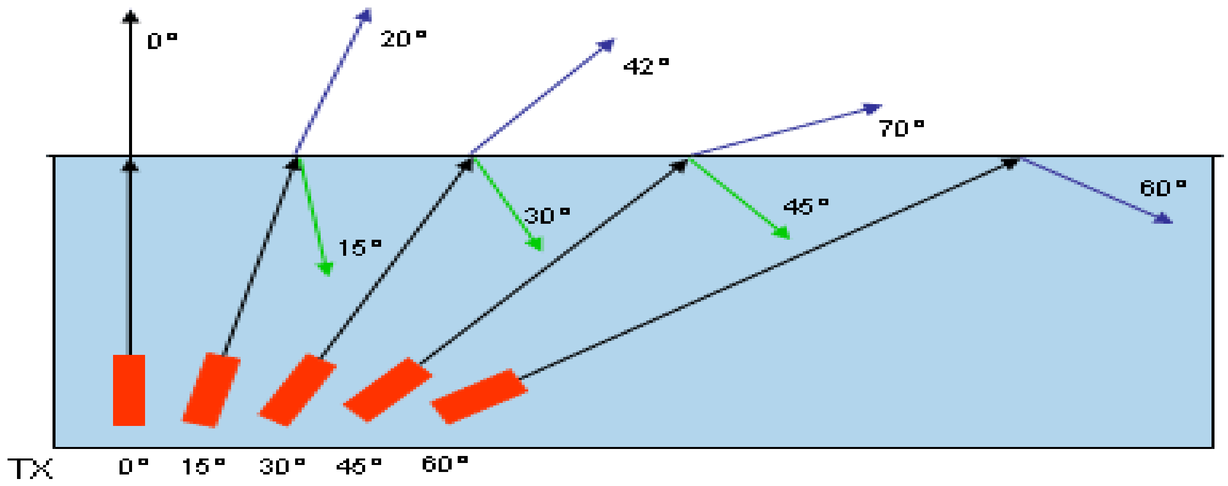
TIR underlies optical guiding in fibers



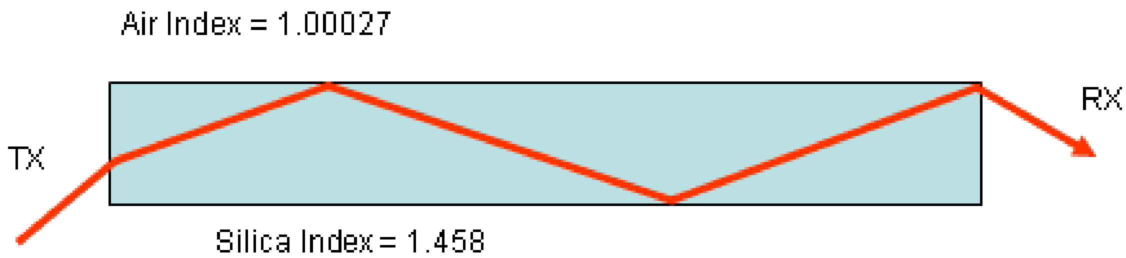

TIR provides confinement for range of input angles

$$\alpha = \sin^{-1}\sqrt{n_1^2 - n_2^2}$$

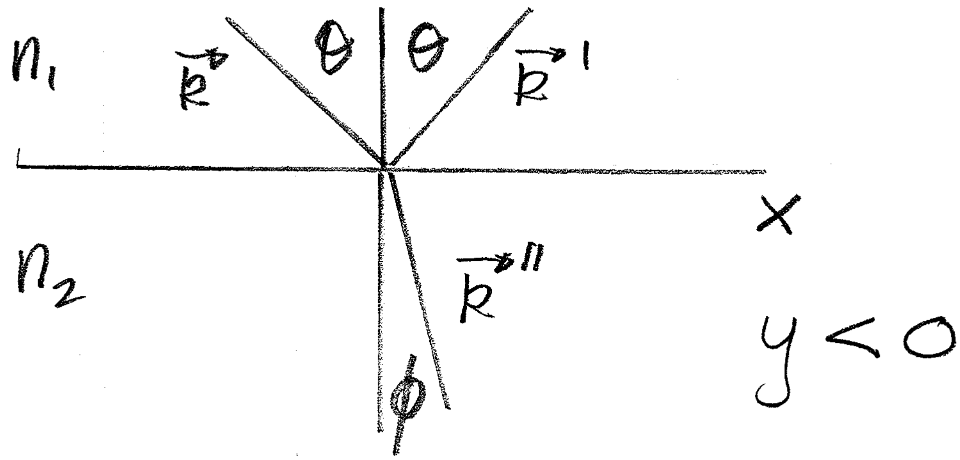
# Total Internal Reflection




# Optical waveguide



## Evanescent waves in TIR $(n < 1)$




Consider TE or s-polarization

$$\vec{E}(\vec{r},\omega) = \hat{z}\,E'' e^{i\vec{k}''\cdot\vec{r}} \qquad y<0$$

$$\vec{k}'' = \frac{n_2\omega}{c}(\sin\phi, -\cos\phi, 0)$$

Snell's law $\sin\phi = \frac{1}{n}\sin\theta$

$$\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\sin^2\theta/n^2}$$

$$= \frac{i}{n}\sqrt{\sin^2\theta - n^2}$$

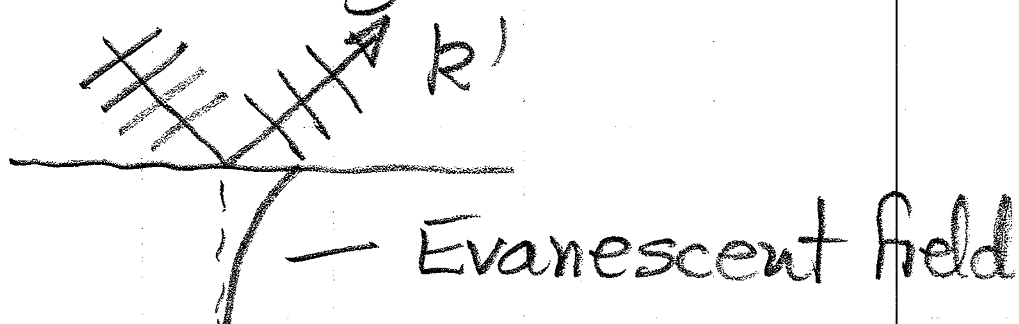
$\cos\phi$ imaginary for TIR, $\theta > \theta_{cr}$

So

$$\vec{E}(\vec{r},\omega) = \hat{z}\, E'' e^{i k_x'' x} e^{\alpha y} \qquad y<0$$

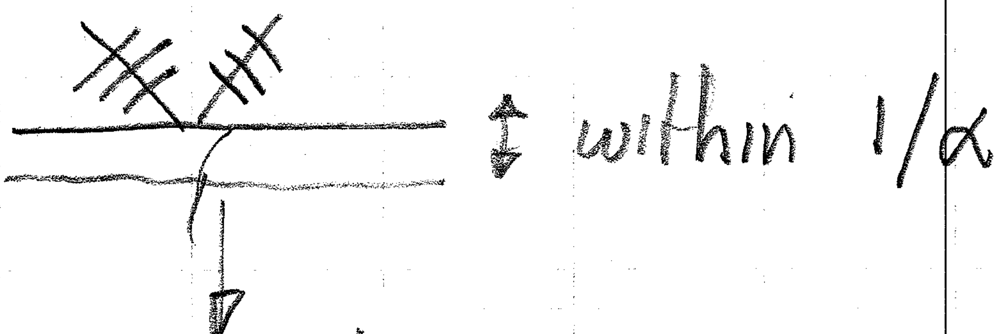
$$\alpha = \frac{n_2 \omega}{c} \frac{\sqrt{\sin^2\theta - n^2}}{n}$$

so the field decays exponentially in medium 2 during TIR.




→ no transmitted energy.

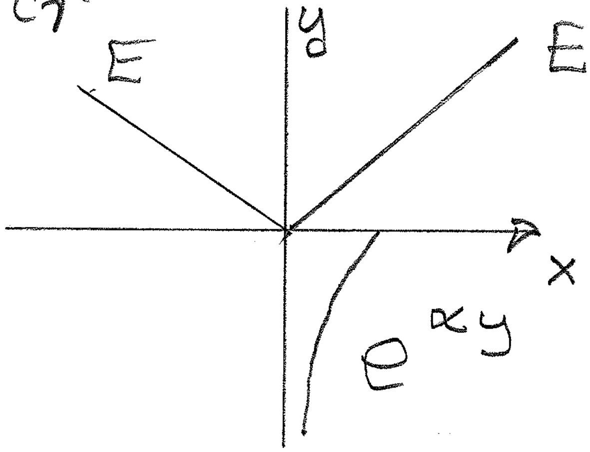
<u>Frustrated TIR</u>




evanescent tail couples into second medium (like QM tunelling)

# Poynting vector for an evanescent field

Following the discussion of TIR in class we get an evanescent field for $\theta > \theta_{cr}$




$\boxed{y < 0}$

where for TE polarization (see notes)

$$\vec{E}(\vec{r},\omega) = \hat{z}\, E(\vec{r},\omega) \qquad (1)$$

$$= \hat{z}\, E'' e^{i k_x'' x} e^{\alpha y} \quad \boxed{y < 0.}$$

The physical evanescent field is then

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left[\vec{E}(\vec{r},\omega)e^{-i\omega t} + c.c\right]$$

$$= \frac{1}{2}\left[\hat{z}E(\vec{r},\omega)e^{-i\omega t} + c.c.\right] \qquad (2)$$

To calculate the Poynting vector we need the associated magnetic field

$$\vec{H}(\vec{r},t) = \frac{1}{2}\left[\vec{H}(\vec{r},\omega)e^{-i\omega t} + c.c\right] \quad (3)$$

in terms of which

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

or using Eqs. (2) & (3) and throwing away fast oscillating terms upon time averaging

$$\langle \vec{S} \rangle_T = \frac{1}{4}\left[\vec{E}(\vec{r},\omega)\times\vec{H}^*(\vec{r},\omega) + c.c\right] \quad (4)$$

We can get $\vec{H}(\vec{r},\omega)$ from Faraday's law (see notes)

$$\vec{H}(\vec{r},\omega) = \frac{1}{i\omega\mu_0}\nabla\times\vec{E}(\vec{r},\omega)$$

with

$$\vec{E}(\vec{r},\omega) = \hat{z}E'' e^{ik_x'' x} e^{\kappa y}$$

which yields

$$\vec{H}(\vec{r},\omega) = \frac{1}{i\omega\mu_0}\left[\alpha\hat{x} - ik_x''\hat{y}\right]E''e^{ik_x''x}e^{\alpha y}$$

We find

$$\vec{E}(\vec{r},\omega)\times\vec{H}^*(\vec{r},\omega) = \frac{|E''|^2 e^{2\alpha y}}{-i\omega\mu_0}\left[\alpha\hat{y} - ik_x''\hat{x}\right]$$

$$\vec{E}^*(\vec{r},\omega)\times\vec{H}(\vec{r},\omega) = \frac{|E''|^2 e^{2\alpha y}}{i\omega\mu_0}\left[\alpha\hat{y} + ik_x''\hat{x}\right]$$

and finally

$$\langle\vec{S}_T\rangle = \frac{1}{2}\frac{k_x''}{\omega\mu_0}e^{2\alpha y}|E''|^2\hat{x}$$

So the time averaged Poynting vector has a component tangential to the interface along the x-axis, but there is no y-component meaning there is no flow of optical energy into the medium.