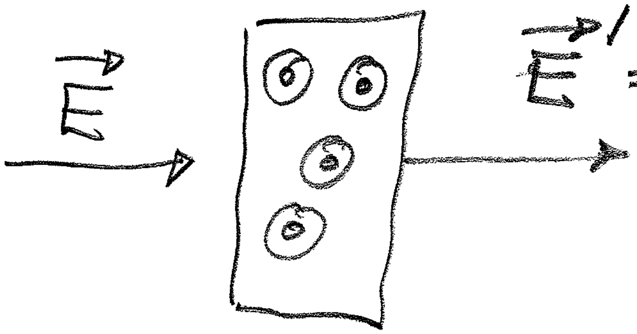
6. Light-matter interaction (Hecht 3,5)

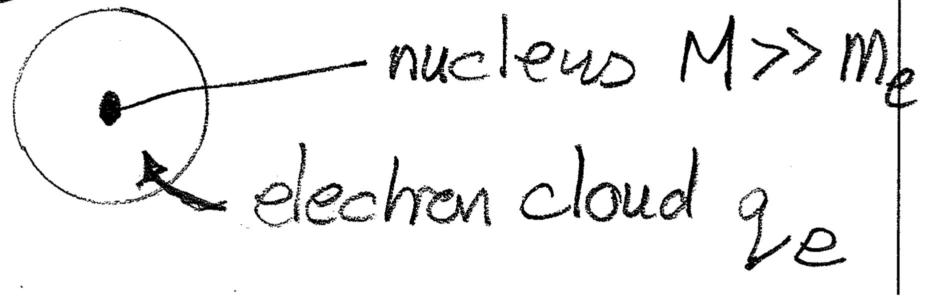
When EM radiation is incident on a material the atoms comprising the medium are excited by the fields, subsequently re-radiate, and modify the field



$$\vec{E}' = \vec{E} + \sum \vec{E}_a$$

The (classical) Lorentz electron oscillator eqn. is an approach to modelling this. Quantum theory gives us a model of an atom as a heavy positively charged and dense nucleus surrouned by a negatively charged cloud representing the light electron

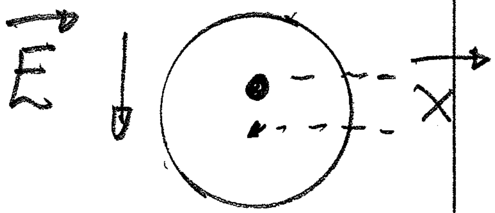
$\vec{X} = 0$



At equilibrium the displacement $\vec{x} = \vec{0}$ between the nucleus (very small) and the charge cloud (radius ~ 1Å) is zero. (the electron cloud ~ sphere).

When an electric field is applied the charge cloud is displaced by the Lorentz force

$$\vec{F}_E = q_e\vec{E}$$



However, there will also by a restoring force due to the Coulomb attraction between the nucleus (charge $-q_e$) and electron (charge $+q_e$), the atom being electrically neutral. We write the Coulomb restoring force as

$$\vec{F}_C = -m_e\omega_0^2\vec{x}$$ (harmonic restoring)

m_e - mass of the electron, ω_0 is an angular frequency to be discussed later. The total force is then

$$\vec{F} = q_e\vec{E} - m_e\omega_0^2\vec{x} = \vec{F}_E + \vec{F}_C.$$

Look at a static field, $\vec{E}$ independent of time. Then at equilibrium, $\vec{F}=0$, giving

$$\vec{x} = \frac{q_e\vec{E}}{m_e\omega_0^2}.$$

and the atom is polarized, the nucleus & electron pulled apart, giving dipole moment

$$\vec{d} = q_e\vec{x} = \frac{q_e^2\vec{E}}{m_e\omega_0^2}.$$

For a medium composed of atoms with density N (atoms per unit vol), the electric polarization is

$$\vec{P} = N\vec{d} = \frac{N q_e^2}{m_e \omega_0^2}\vec{E}$$

which is dipole moment/Volume. The polarization is a measure of the excitation of the atoms by the incident field $\vec{E}$, and $\vec{P}$ determines how the excited atoms feedback onto the incident field.

The electric displacement is a combination of $\vec{E}$ and $\vec{P}$ that reflects the back-action

$$\vec{D} = \epsilon\vec{E} = \epsilon_0 n^2 \vec{E}$$

$$= \epsilon_0\vec{E} + \vec{P} \quad \left(= \epsilon_0\vec{E} + \epsilon_0 \sum{}' \vec{E}_a \right)$$

with n the refractive-index of the medium. For our example of

a static field $\vec{P} = (Nq_e^2/m_e\omega_o^2)\vec{E}$,
and

$$\boxed{n^2 = 1 + \frac{Nq_e^2}{\epsilon_o m_e \omega_o^2}}$$

this is the <u>refractive-index</u> for <u>static electric fields</u>, and the dielectric constant is

$$\epsilon = \epsilon_o n^2 = \epsilon_o + \frac{Nq_e^2}{m_e\omega_o^2}$$

the optical properties, ϵ & n, are thus modified by the presence of the medium. Medium properties are N & ω_o - what physically is ω_o? More later.

Next we need a similar treatment for non-static light fields, $\omega \sim 10^{16}$ rad s^{-1}! (static case $\omega = 0$!)

Refractive-index and dispersion

To proceed return to the Newton's equation for the charge displacement

$$\vec{F} = m_e \vec{a} = m_e \frac{d^2 \vec{x}}{dt^2} = q_e \vec{E} - m_e \omega_0^2 \vec{x}.$$

or

$$\boxed{m_e \frac{d^2 \vec{x}}{dt^2} = q_e \vec{E}(t) - m_e \omega_0^2 \vec{x}(t)}$$

Consider a monochromatic field

$$\left.\begin{array}{l} \vec{E}(t) = \hat{e} E_0 \cos(\omega t) \\ \text{and try} \\ \vec{x}(t) = \hat{e}\, x_0 \cos(\omega t) \end{array}\right\}$$

then

$$-m_e \omega^2 \cancel{\hat{e}}\, x_0 \cancel{\cos(\omega t)} = q_e \cancel{\hat{e}} E_0 \cancel{\cos(\omega t)} - m_e \omega_0^2 \cancel{\hat{e}}\, x_0 \cancel{\cos(\omega t)}$$

This yields

$$x_0 = \frac{(q_e/m_e)}{(\omega_0^2 - \omega^2)} E_0$$

and

$$\boxed{\vec{x}(t) = \frac{(q_e/m_e)}{(\omega_0^2 - \omega^2)} \vec{E}(t).}$$

The <u>induced dipole moment/atom</u> is then

$$\vec{d}(t) = q_e \vec{x}(t) = \frac{(q_e^2/m_e)}{(\omega_0^2 - \omega^2)} \vec{E}(t)$$

so if $\omega_0 > \omega$, $\vec{d}(t)$ is in phase with $\vec{E}(t)$.

This leads to the <u>electric polarization</u> for a medium of density N of atoms

$$\vec{P}(t) = N\vec{d}(t) = \frac{(Nq_e^2/m_e)}{(\omega_0^2 - \omega^2)} \vec{E}(t)$$

and, finally, for the <u>optical displacement</u>.

$$\vec{D}(t) = \epsilon_0 \vec{E}(t) + \vec{P}(t)$$

$$= \epsilon_0 \vec{E}(t) + \frac{(Nq_e^2/m_e)}{(\omega_0^2 - \omega^2)} \vec{E}(t)$$

$$= \epsilon_0 \left[1 + \frac{(Nq_e^2/\epsilon_0 m_e)}{(\omega_0^2 - \omega^2)} \right] \vec{E}(t)$$

$$= \epsilon_0 n^2(\omega) \vec{E}(t) = \epsilon(\omega) \vec{E}(t).$$

We then find for the frequency <u>dependent refractive-index $n(\omega)$</u>

$$\boxed{n^2(\omega) = 1 + \frac{(Nq_e^2/\epsilon_0 m_e)}{(\omega_0^2 - \omega^2)}}$$

Note that this reduces to the previous result for the static case $\omega = 0$.

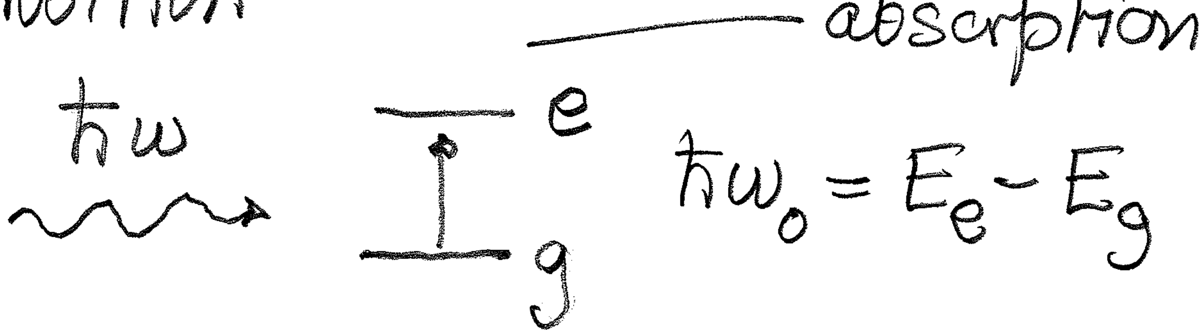
The frequency dependence of $n(\omega)$ is called <u>optical dispersion</u>

Notes on the result (Hecht Eq. 3.70)

$$n^2(\omega) = 1 + \left(\frac{Nq_e^2}{\epsilon_0 m_e}\right)\frac{1}{(\omega_0^2 - \omega^2)}$$

- If $N=0$, no medium, $n(\omega)=1$, eg. vacuum.
- For $\omega \rightarrow \omega_0$, $n(\omega) \rightarrow \pm\infty$! (plot)

Corresponds to resonance between the field energy/photon and atomic transition

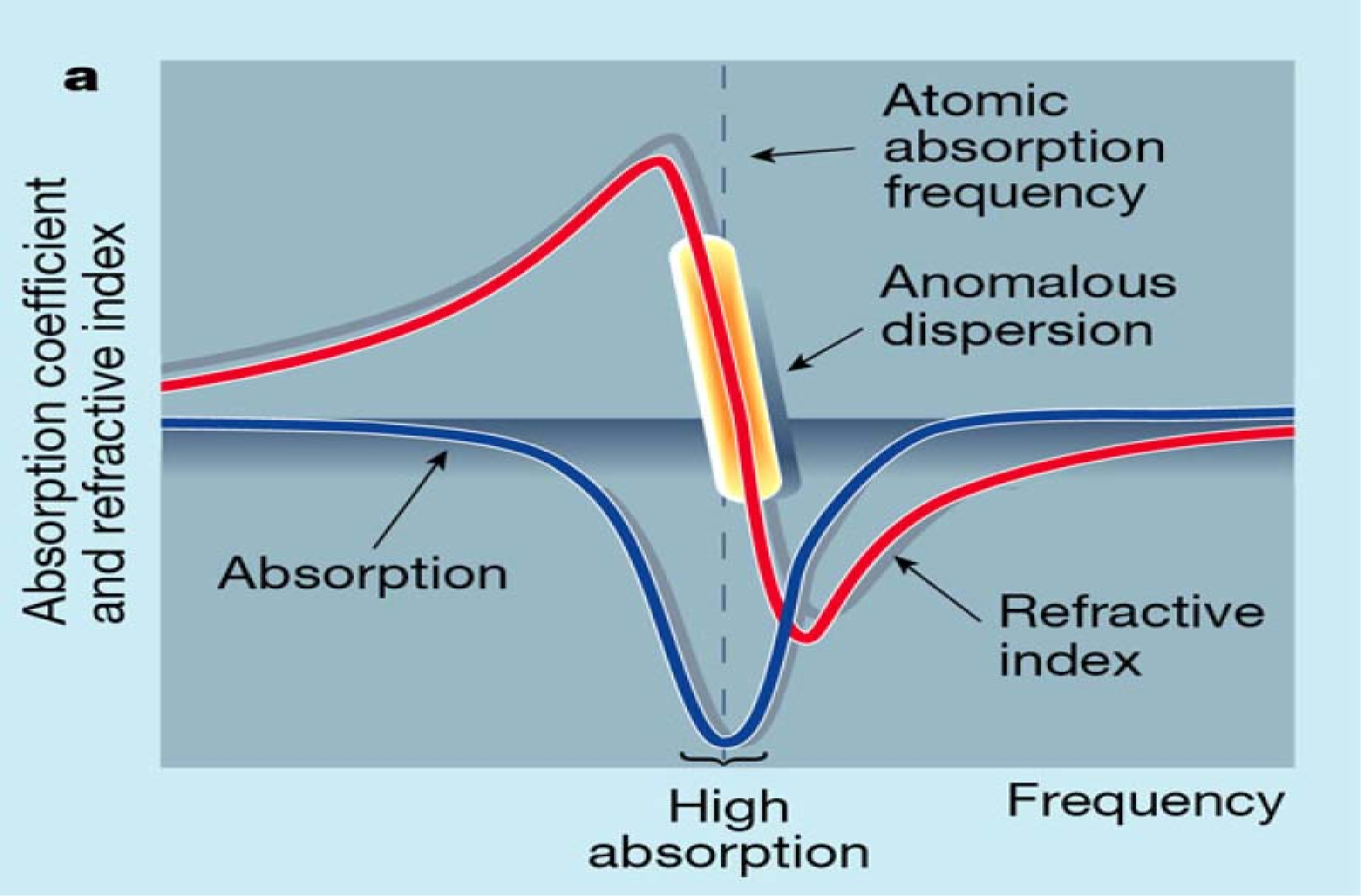


$$\hbar\omega_0 = E_e - E_g$$

Strong absorption when $\hbar\omega = \hbar\omega_0$, current theory invalid.

ω_0 is an atomic property related to energy level spacing Need QM, here a parameter

Optical Dispersion



- Theory is good for dielectrics when $\omega_0 >> \omega$, then little absorption.

- In general an atom has many levels and possible transitions labelled j

$$n^2(\omega) = 1 + \left(\frac{N q_e^2}{\epsilon_0 m_e}\right) \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2)}$$

f_j - <u>oscillator strengths</u> of transitions.

- Using $\omega = 2\pi c/\lambda$, with λ the vacuum wavelength, and assuming $\omega_{0j} >> \omega$ for all j, one obtains the approx.

$$n^2(\lambda) = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + \ldots$$

This is called the <u>Cauchy formula</u> for the refractive-index of <u>transparent dielectric media</u>.

Birefringence in crystals (Hecht, 8.4)

In this class we shall be concerned with transparent dielectric in the spectral region $\omega \ll \omega_o$, so

$$n^2(\omega) \simeq 1 + \frac{Nq_e^2}{\epsilon_o m_e \omega_o^2} \sim \text{const.}$$

So far we have tacitly assumed a gas of atoms or an isotropic medium

- We have assumed our atom would respond the same, with ω_o independent of the direction of the applied field $\vec{E}$

$$\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}$$

For atoms in crystals the anisotropic crystal environment can mean ω_o can depend on the direction of the applied field wrt crystal axes x, y, z, leading to $\omega_{ox}, \omega_{oy}, \omega_{oz}$. Then the refractive-index also depends on the direction of the applied field n_x, n_y, n_z.

$$n_x^2 \simeq 1 + \frac{N q_e^2}{\epsilon_0 m_e \omega_{ox}^2} \quad , \text{ etc.}$$

This is called <u>birefringence</u> and it occurs in <u>anisotropic crystals</u> where the atoms are arranged in <u>unit cells</u> that have low symmetry.

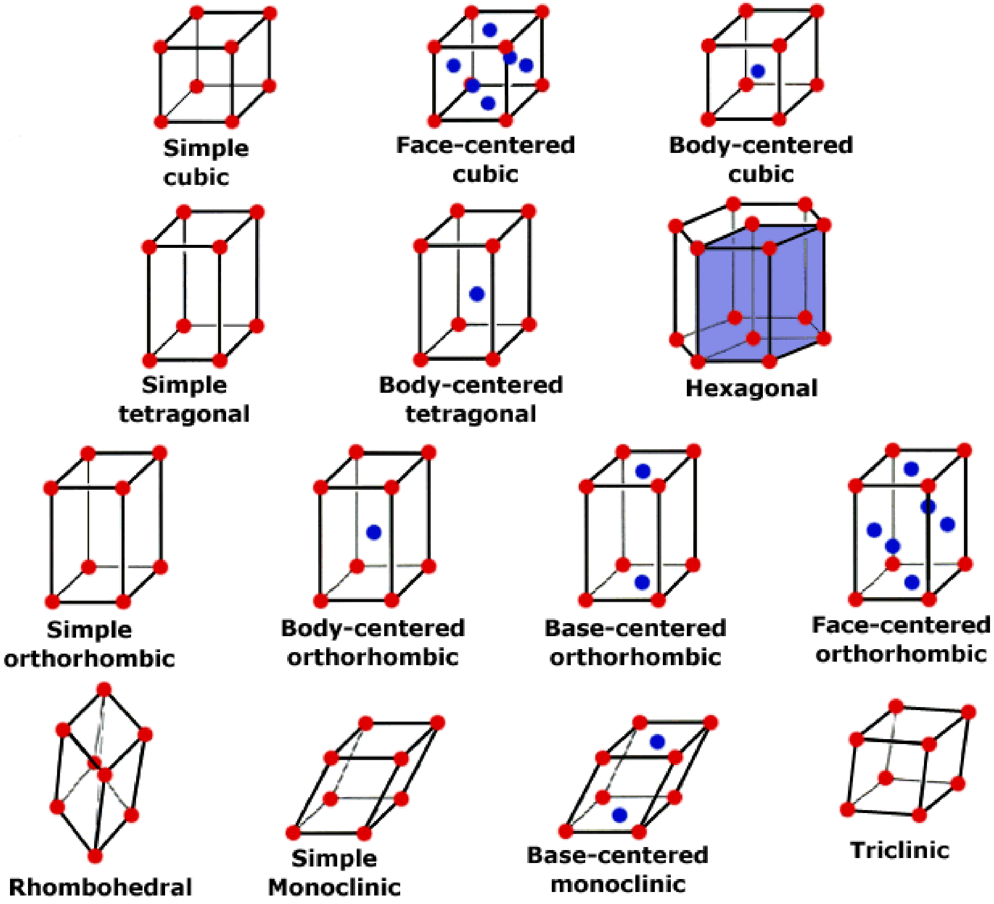
<u>Cubic crystals</u> are <u>isotropic</u>

$$n_x = n_y = n_z$$

<u>Uniaxial</u> crystals have two equal one different

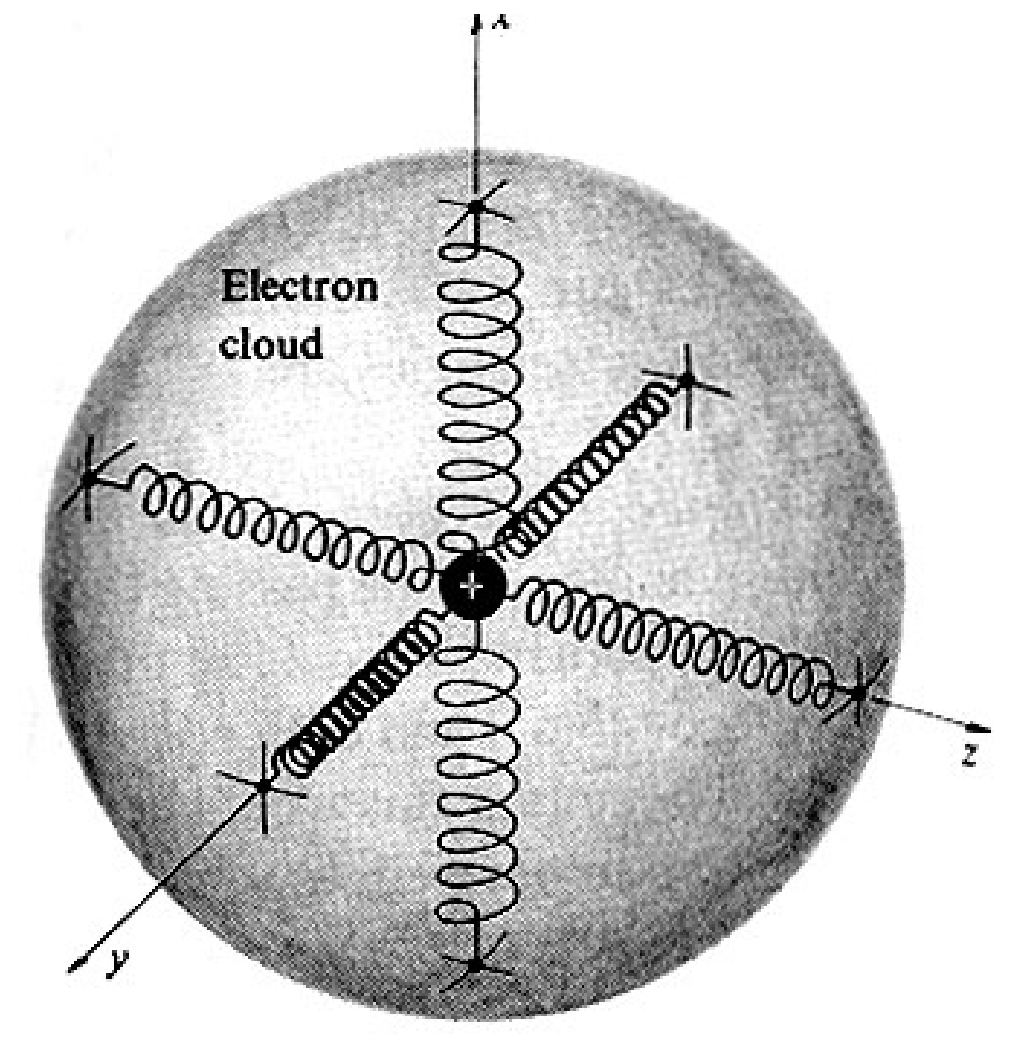
<u>Biaxial</u> $n_x \neq n_y \neq n_z \neq n_x$

Crystal unit cells – Bravais lattices



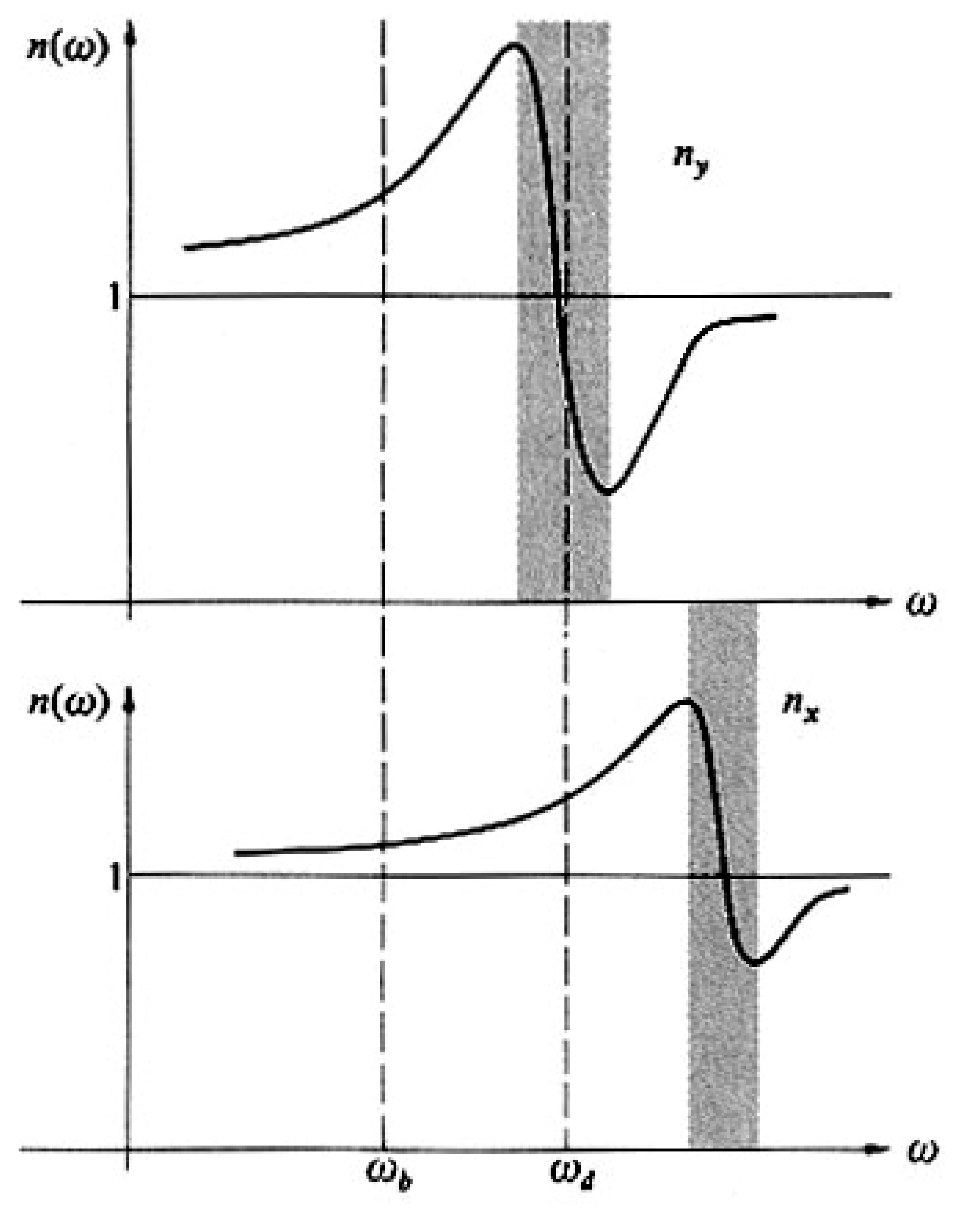
Birefringence

The molecular "spring constant" can be different for different directions.

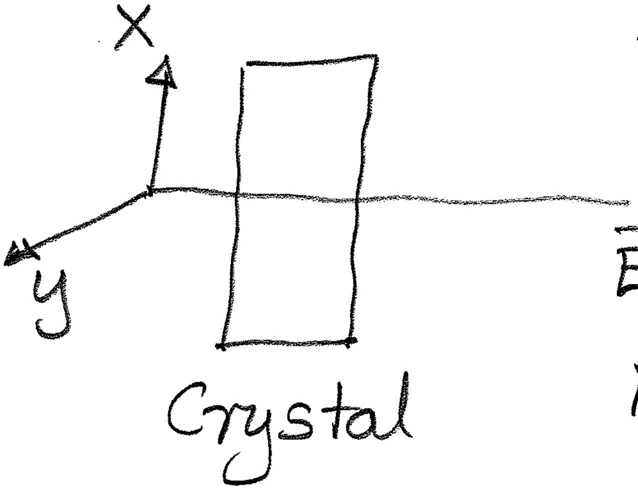


Birefringence

The x- and y-polarizations can see different refractive index curves.



To summarize, <u>optical birefringence</u> in crystals is an effect in which the refractive-index experienced by a light field depends on the vector direction of it's electric field



$\vec{E} \propto \hat{i}$

$n = n_x$

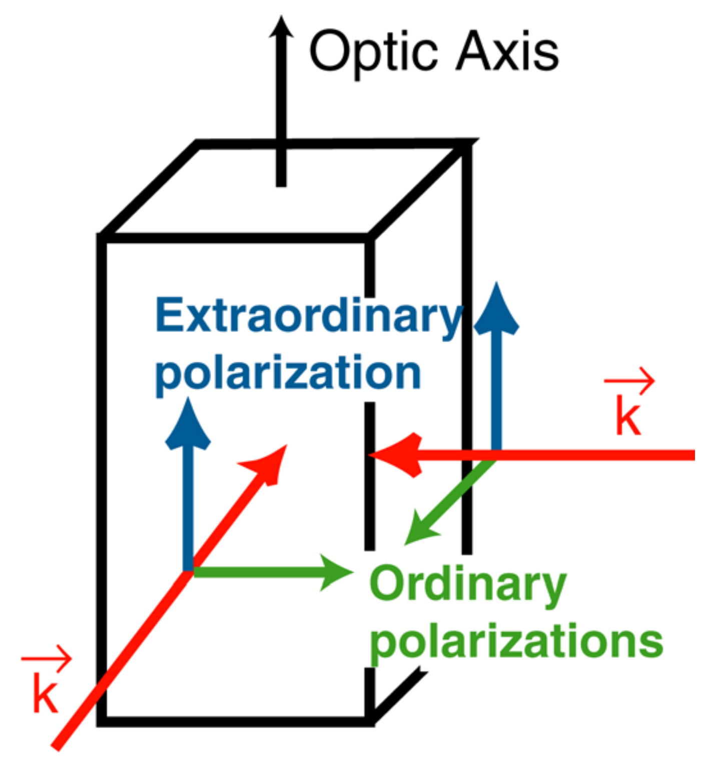
$\vec{E} \propto \hat{j}$

$n = n_y$.

so the light velocity is different also $V_x = c/n_x$, $V_y = c/n_y$. This causes <u>double refraction</u>

We shall use the idea of optical birefringence when considering <u>polarization.</u>

Uniaxial crystals have an optic axis

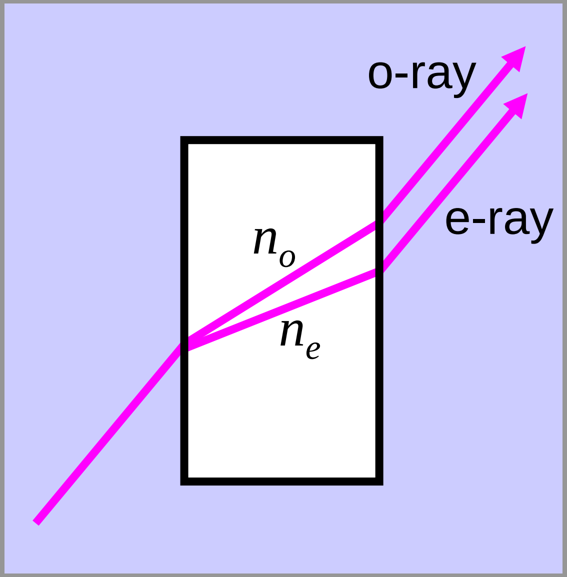


Uniaxial crystals have one refractive index for light polarized along the **optic axis** (n_e) and another for light polarized in either of the two directions perpendicular to it (n_o).

Light polarized along the optic axis is called the **extraordinary** ray, and light polarized perpendicular to it is called the **ordinary** ray. These polarization directions are the **crystal principal axes**.

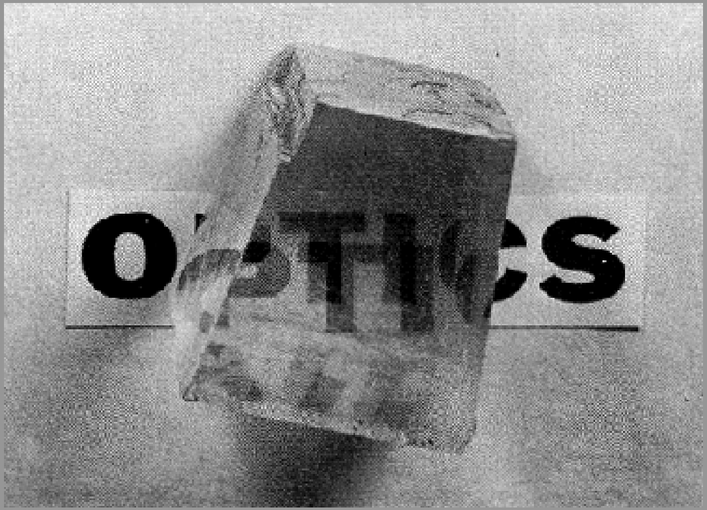
Light with any other polarization must be broken down into its ordinary and extraordinary components, considered individually, and recombined afterward.

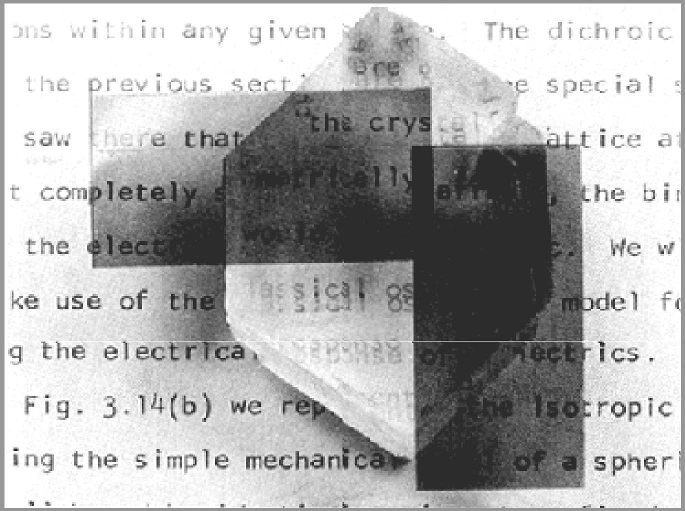
Birefringence can separate the two polarizations into separate beams



Due to Snell's Law, light of different polarizations will bend by different amounts at an interface.

Calcite





Calcite is one of the most birefringent materials known.

Birefringent Materials

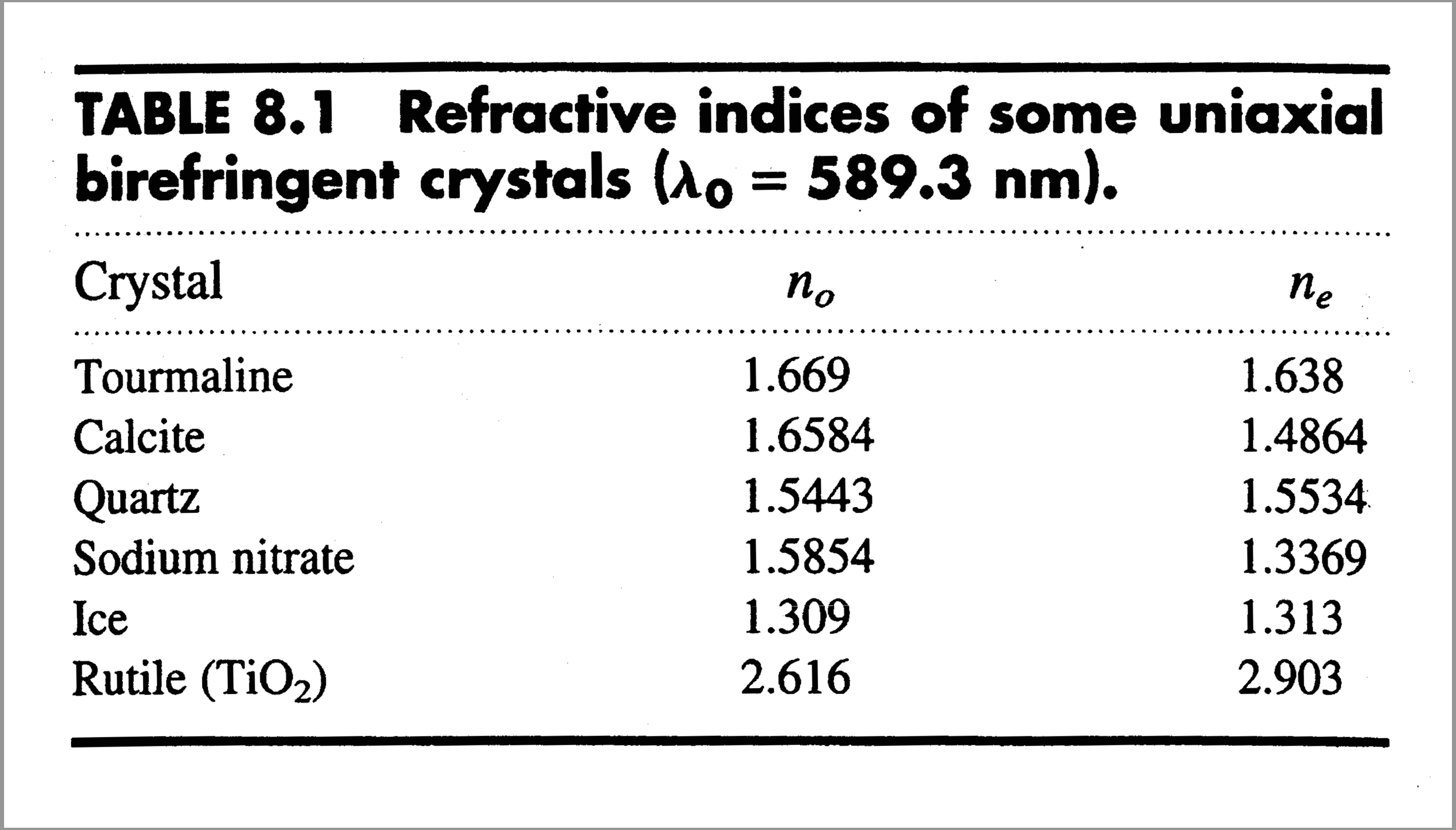


TABLE 8.1 Refractive indices of some uniaxial birefringent crystals ($\lambda_0 = 589.3$ nm).

Crystal	n_o	n_e
Tourmaline	1.669	1.638
Calcite	1.6584	1.4864
Quartz	1.5443	1.5534
Sodium nitrate	1.5854	1.3369
Ice	1.309	1.313
Rutile (TiO_2)	2.616	2.903

Harmonic EM waves.

We have now finished the part of the class aimed at understanding the harmonic plane-wave solutions of Maxwell's equations. Next we shall turn to how those solutions are employed in basic optical situations, eg. interface, Fabry-Perot, ... First we here revisit some notations for the fields and expand upon some previous ideas.

Notation

We write the harmonic solution as

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

and $\vec{E}$, $\vec{B}$ & $\vec{k}$ form a RH set, and $w = vk$, $v = c/n$. Using Euler's formula we can write, eg.

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left[\vec{E}_0 e^{i\varepsilon} e^{i(\vec{k}\cdot\vec{r}-wt)} + c.c\right]$$

$$= Re\left[\vec{E}_0 e^{i\varepsilon} e^{i(\vec{k}\cdot\vec{r}-wt)}\right]$$

Often we write the fields in complex representation

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i\varepsilon} e^{i(\vec{k}\cdot\vec{r}-wt)}$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{i\varepsilon} e^{i(\vec{k}\cdot\vec{r}-wt)}$$

with the understanding that one must take the real part.

We shall employ the notation

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{k},w) e^{i(\vec{k}\cdot\vec{r}-wt)}$$

$$\vec{B}(\vec{r},t) = \vec{B}(\vec{k},w) e^{i(\vec{k}\cdot\vec{r}-wt)}$$

where

$$\vec{E}(\vec{k},\omega) = \vec{E}_0 e^{i\varepsilon}, \quad \vec{B}(\vec{k},\omega) = \vec{B}_0 e^{i\varepsilon}$$

are complex amplitudes for the electric & magnetic harmonic field solutions. Recalling the following equivalences for harmonic solns.

$$\nabla e^{i(\vec{k}\cdot\vec{r}-\omega t)} \equiv i\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \equiv -i\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

then we may replace $\nabla \rightarrow i\vec{k}$, and $\partial/\partial t = -i\omega$. Consider Faraday's law for our harmonic solutions

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}}{\partial t}$$

becomes

$$i\vec{k}\times\vec{E}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= +i\omega\vec{B}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

or cancelling the exponentials & i

$$\boxed{\vec{k}\times\vec{E}(\vec{k},\omega) = \omega\vec{B}(\vec{k},\omega)}$$

$\vec{B}$ is the <u>magnetic flux density</u>, and it is often written as $\vec{B} = \mu_0\vec{H}$, with $\vec{H}$ the <u>magnetic intensity</u> (ampere/m). With this we have

$$\boxed{\begin{aligned}\vec{H}(\vec{k},\omega) &= \frac{\vec{B}(\vec{k},\omega)}{\mu_0}\\ &\Rightarrow \frac{1}{\omega\mu_0}\vec{k}\times\vec{E}(\vec{k},\omega)\end{aligned}}$$

Thus Faraday's law provides a relation between the complex field amplitudes.- If I know $\vec{E}(\vec{k},\omega)$, ω, $\vec{k}$ I can get $\vec{B}(\vec{k},\omega)$ or $\vec{H}(\vec{k},\omega)$. I have introduced $\vec{H}$ as Fowles uses this form.

Helmholtz equation

The complex fields are also sometimes written as

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i\varepsilon} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \vec{E}(\vec{r},\omega) e^{-i\omega t}$$

with

$$\vec{E}(\vec{r},\omega) = \vec{E}_0 e^{i\varepsilon} e^{i\vec{k}\cdot\vec{r}}$$

What equation does $\vec{E}(\vec{r},\omega)$ obey?

Start from the wave equation

$$\left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = 0$$

$$= \left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},\omega)e^{-i\omega t}$$

$$= e^{-i\omega t}\left[\nabla^2 + k^2\right]\vec{E}(\vec{r},\omega) = 0$$

giving the <u>Helmholtz equation</u>

$$\left[\nabla^2 + k^2\right]\vec{E}(\vec{r},\omega) = 0$$

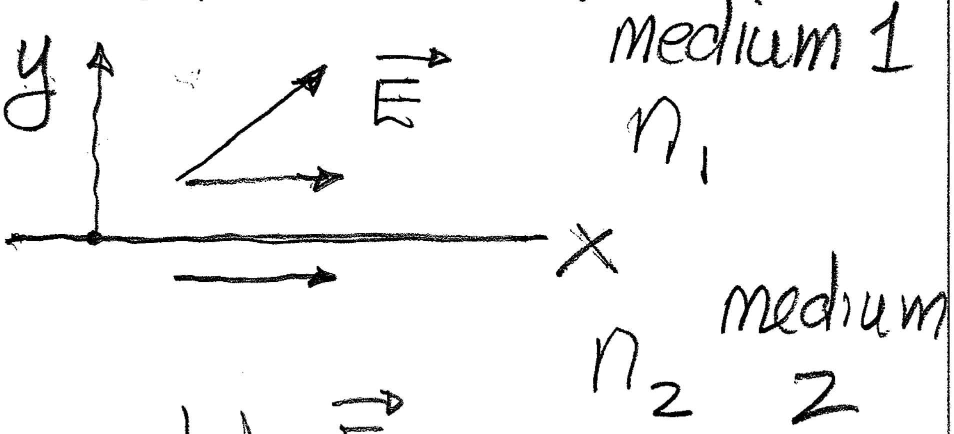
with $k = n\omega/c$. You can check that a valid solution of this equation is

$$\vec{E}(\vec{r},\omega) = \vec{E}_0 e^{i\varepsilon} e^{i\vec{k}\cdot\vec{r}}$$

$$= \vec{E}(\vec{k},\omega)e^{i\vec{k}\cdot\vec{r}}$$

Boundary conditions

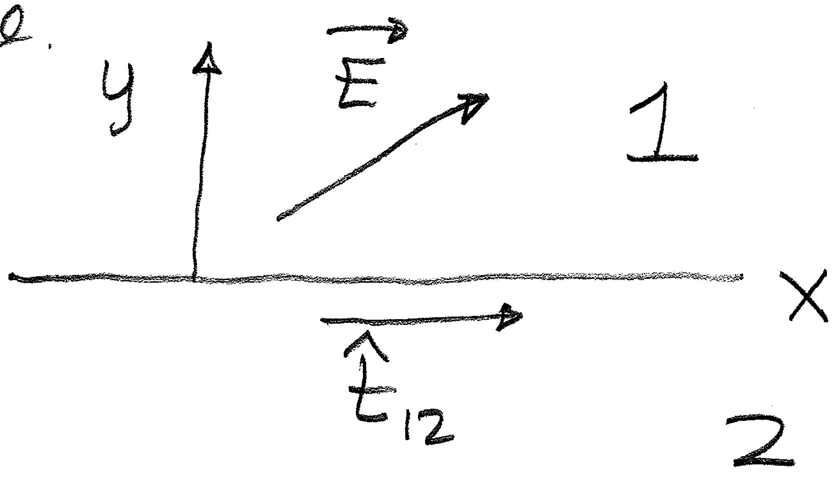
In the next two sections we shall consider problems involving the interface between media of differing refractive-index.



How should we relate $\vec{E}$ on either side of the boundary, that we take to be the (x-z) plane?

It turns of that the components of $\vec{E}$ & $\vec{H}$ (or $\vec{B}$) tangential to the boundary MUST be continuous across the boundary. This is stated without proof.

How can we obtain the component of the field tangential to the interface.



$\hat{t}_{12}$ is a vector tangential to the boundary, eg $\hat{t}_{12} = \hat{\imath} \equiv \hat{x}$ or

$\hat{t}_{12} = \hat{k} \equiv \hat{z}$

$\hat{\imath}, \hat{\jmath}, \hat{k} \equiv \hat{x}, \hat{y}, \hat{z}$. Then the component of $\vec{E}$ tangential to the interface is

$$\boxed{E = \vec{E} \cdot \hat{t}_{12}}$$

eg.

$$E_x = \hat{x} \cdot \vec{E} = \hat{\imath} \cdot \vec{E} \text{ etc.}$$