5. EM Waves – Maxwell's eqns.

We now have Maxwell's eqns. in integral form, and we shall convert them to differential form (Hecht 3.2, App. 1)

- Gauss' law

For a dielectric medium with no free charges the electric & magnetic fluxes are zero

$$\underbrace{\oiint_A \vec{E}\cdot d\vec{S}}_{} = \iiint_V dV\, \underbrace{\nabla\cdot\vec{E}}_{} = 0$$

integral form

→ Gauss' theorem

This is obviously satisfied if

diff. form

$$\boxed{\nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.}$$

This is the differential form of Gauss' law for the electric field.

Similarly, Gauss' law for the magnetic field is in differential form

$$\boxed{\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 .}$$

- Faraday's law

We have.

$$\oint_C \vec{E} \cdot d\vec{l} \overset{\text{Stokes' theorem}}{=} \iint_A (\nabla \times \vec{E}) \cdot d\vec{S}$$

$$= - \iint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

or

$$\iint_A \underbrace{\left[\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right]}_{=0} \cdot d\vec{S} = 0$$

One way for this to be satisfied is if the term in square brackets is set to zero, giving the <u>differential form of Faraday's law</u>

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

- PDE relating curl of $\vec{E}$ with time deriv. of $\vec{B}$
- For <u>static case</u> $\nabla \times \vec{E} = 0$.

- <u>Ampere's law</u>

We have ⟶ Stokes' theorem

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_A (\nabla \times \vec{B}) \cdot d\vec{S}$$

$$= \mu_0 \epsilon \iint_A \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}$$

or

$$\iint_A \left[\nabla \times \vec{B} - \mu_0 \epsilon \frac{\partial \vec{E}}{\partial t} \right] = 0$$

If we set the term in square brackets to zero we get Ampere's law in differential form

$$\boxed{\nabla \times \vec{B} = \mu_0 \epsilon \frac{\partial \vec{E}}{\partial t}}$$

- PDE relating curl of $\vec{B}$ to time deriv. of $\vec{E}$
- For static case $\nabla \times \vec{B} = 0$.

In summary Maxwell's eqns in diff. form are for a dielectric medium

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday)}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{(Ampere)}$$

$$\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0 \quad \text{(Gauss)}.$$

Next we obtain wave eqns for the EM fields.

Wave equation

Maxwell's equation do not yet expose that light fields are a wave phenomena. To do this take the curl of Faraday's law

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$ use Ampere's law

$$= -\mu_0 \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

where $v^2 = 1/(\mu_0 \epsilon)$. Now

$$\nabla \times \nabla \times \vec{E} = \nabla(\underbrace{\nabla \cdot \vec{E}}_{=0 \text{ by Gauss' law.}}) - \nabla^2 \vec{E}$$

thus we obtain

$$\boxed{\nabla^2 \vec{E} = \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2}}$$

that is a 3D wave equation for the vector electric field! The wave eqn applies for each vector component E_x, E_y, E_z, separately, eg.

$$\boxed{\nabla^2 E_x = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2}}$$

$E_x(\vec{r}, t)$ being a scalar field....
However, the 3 scalar components are inter related via Gauss' law

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.$$

If a solution for $\vec{E}$ is known we can calculate $\partial \vec{B}/\partial t$ from Faraday's law

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

In optics we typically focus our attention on the electric field vector, though one can also obtain a wave equation for the magnetic field.

The velocity v of EM field propagation in a dielectric medium is given by

$$V = \frac{1}{\sqrt{\mu_0 \epsilon}} = \frac{1}{\sqrt{\mu_0 \epsilon_0 K_E}}$$

$$= \frac{c}{n}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0} = 3 \times 10^8 \, m/s$ is the speed of light in vacuum, and $n = \sqrt{K_E}$ is the refractive-index of the dielectric medium.

- Transverse EM waves.

We start from the harmonic solution for the electric field (complex rep.)

$$\vec{E}(\vec{r},t) = \hat{e} A e^{i(\vec{k}\cdot\vec{r} - \omega t + \epsilon)}$$

- $\hat{e}$ – unit vector in direction of E-field
- $\vec{k}$ – propagation vector
- ω – frequency
- A & ϵ – amplitude & phase

As always to get the physical field we take the real part

$$\vec{E}(\vec{r},t) = \hat{e} A \cos(\vec{k}\cdot\vec{r} - \omega t + \epsilon)$$

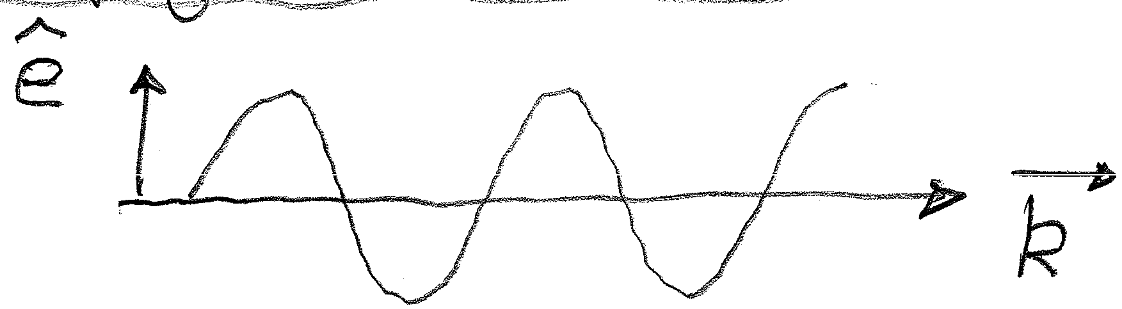
Describes an electric field with vector direction $\hat{e}$ propagating in the direction of $\vec{k}$.

What can we learn from Maxwell's equations? Try Gauss' law

$$\nabla \cdot \vec{E} = \nabla \cdot \left(\hat{e} A e^{i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)}\right)$$

$$= i\vec{k}\cdot\hat{e}\, A e^{i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)} = 0$$

$\boxed{\text{This implies } \hat{e} \perp \vec{k}\text{, the electric field oscillates in a vector direction } \hat{e} \text{ transverse to the direction of propagation } \vec{k}.}$



Eg. For propagation along the x-axis
$\vec{k} = k\hat{\imath}$, $\hat{e} = \hat{\jmath}$ or $\hat{k}$, for example

Take a specific example, $\vec{k} = k\hat{i}$, $\hat{e} = \hat{j}$, so in the <u>real representation</u>

peak field $A = E_{0y}$

$$\vec{E}(\vec{r},t) = \hat{j}\,E_y(x,t)$$

$$= \hat{j}\,A\cos(kx-\omega t+\varepsilon)$$

$$= \hat{j}\,E_{0y}\cos(kx-\omega t+\varepsilon).$$

Electric field is:

- propagating along the x-axis
- <u>Linearly polarized</u> along y-axis
- Plane-wave, independent of z, y.

Substitute in the wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \hat{j}\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)E_y$$

$$= 0$$

or

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)E_y(x,t)$$

Like <u>1D scalar wave eqn.</u> for a <u>transverse field</u>! Thus we get for our solution $E_y(x,t)$

$$\boxed{v = \frac{c}{n} = \frac{\omega}{k}.}$$

— <u>Electric field is a transverse wave</u>.

How about the magnetic field? Use Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\nabla \times \vec{E} = \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$+ \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

But we have $E_x = E_z = 0$, $E_y(x,t)$, so

$$\nabla \times \vec{E} = \hat{k}\,\frac{\partial E_y}{\partial x} = -\frac{\partial \vec{B}}{\partial t}$$

This means the magnetic field must be directed along the z-axis

$$\boxed{\vec{B}(\vec{r},t) = \hat{k}\,B_z(x,t)}$$

with

$$\nabla \times \vec{E} = \hat{k}\,\frac{\partial E_y}{\partial x} = -\frac{\partial \vec{B}}{\partial t} = -\hat{k}\,\frac{\partial B_z}{\partial t}$$

or

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \qquad \text{(Hecht 3.27)}$$

For our harmonic plane-wave soln.

$$\left.\begin{aligned} E_y(x,t) &= E_{oy}\cos(kx-\omega t+\varepsilon) \\ \frac{\partial E_y}{\partial x} &= -kE_{oy}\sin(kx-\omega t+\varepsilon) \end{aligned}\right\}$$

Then use

$$B_z(x,t) = -\int dt \left(\frac{\partial E_y}{\partial x} \right)$$

$$= \left(\frac{k}{\omega} \right) E_{oy} \cos(kx - \omega t + \varepsilon)$$

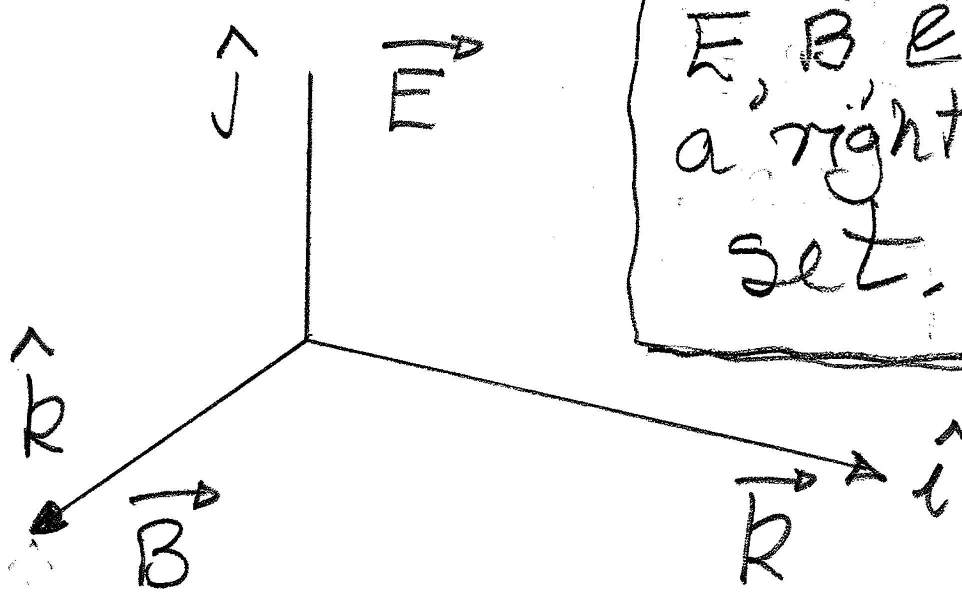
$$= B_{oz} \cos(kx - \omega t + \varepsilon)$$

where $B_{oz} = (k/\omega) E_{oy} = (1/v) E_{oy}$,

or

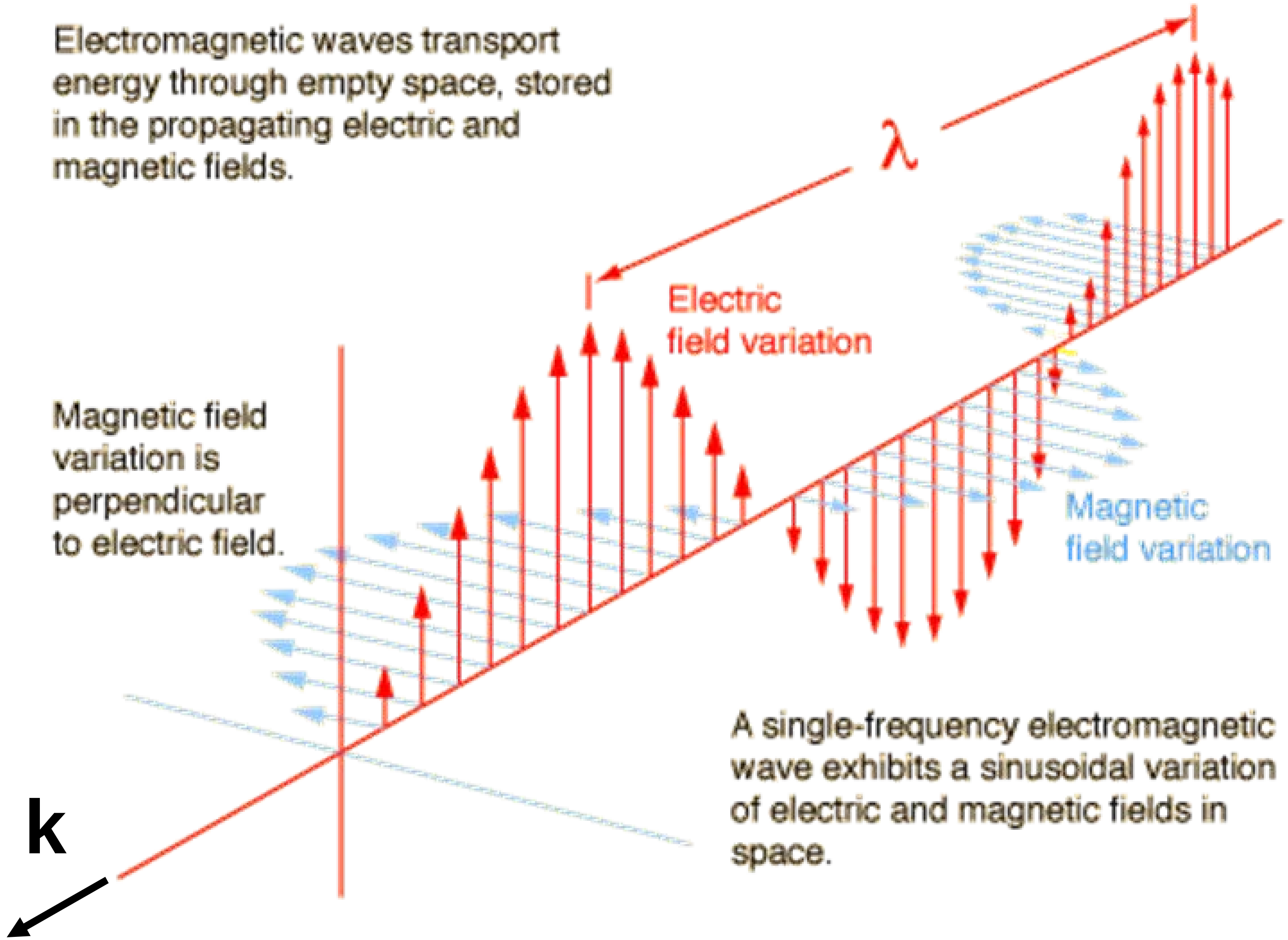
$$\boxed{E_{oy} = v B_{oz}}$$

$\vec{E}$, $\vec{B}$ & $\vec{k}$ form a right-handed set.



See Hecht Figs. 3.13, 3.14

Light is a transverse electromagnetic wave



Summary

Consider a harmonic EM field propagating along the direction given by the propagation vector $\vec{k}$

$$\vec{E}(\vec{r},t) = \hat{e}\,E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

$$\vec{B}(\vec{r},t) = \hat{b}\,B_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

- Gauss's laws tell us

$$\boxed{\hat{e}\cdot\vec{k} = \hat{b}\cdot\vec{k} = 0}$$

→ field is transverse

- Faraday's (or Ampere's) law tells us

$$E_0 = v B_0 = (c/n)\,B_0$$

and $\hat{e}\cdot\hat{b} = 0$, $\vec{E} \perp \vec{B}$, $\hat{e}\times\hat{b} \parallel \vec{k}$

- Wave equation tells us

$$\boxed{\omega = v k}$$

Wavefronts and rays

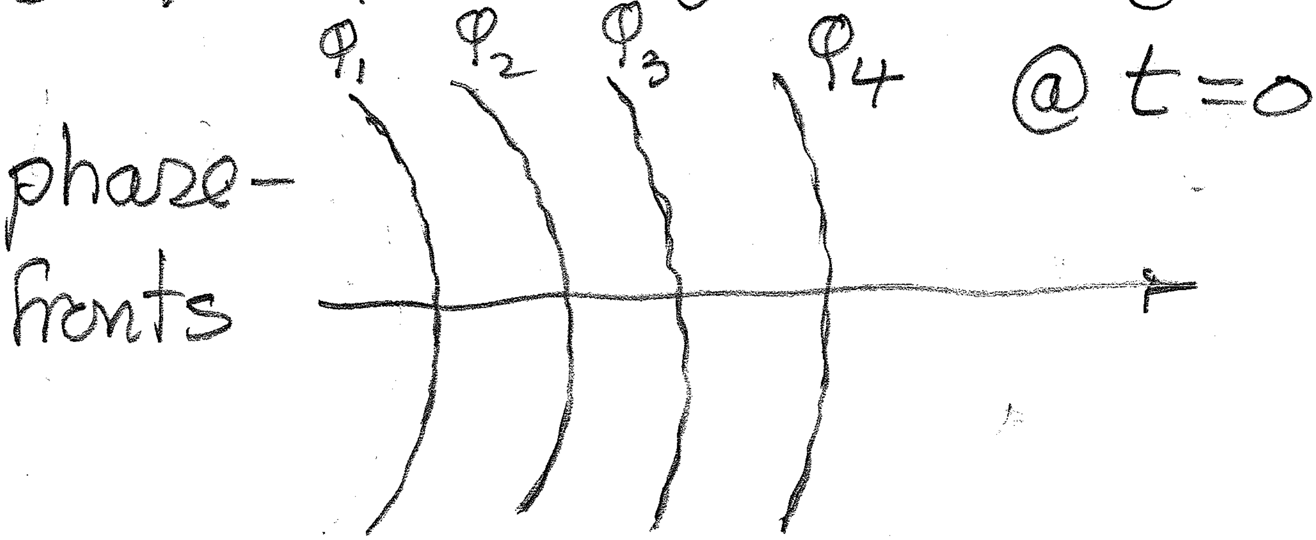
Consider the harmonic electric field

$$\vec{E}(\vec{r},t) = A\hat{e}\, e^{i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)}$$

$$= \vec{E}_0 e^{i\varphi(\vec{r},t)}, \quad \vec{E}_0 = A\hat{e}\, e^{i\varepsilon}$$

with the field $\varphi(\vec{r},t) = (\vec{k}\cdot\vec{r} - \omega t)$ representing the <u>phase of the electric field</u>.

<u>Phase-fronts</u> or <u>wavefronts</u> are lines of equal phase of $\varphi(\vec{r},t)$, eg



they are like isotherms for a temp. field, so they are "isophase" lines.

Geometric optical rays are paths that are everywhere orthogonal to the phase-fronts

The ray direction thus follows from

$$\vec{n}(\vec{r},t) \propto \nabla\varphi(\vec{r},t)$$

For our harmonic solution

$$\vec{n} \propto \vec{k}$$

so the rays are parallel to $\vec{k}$. (see next page).

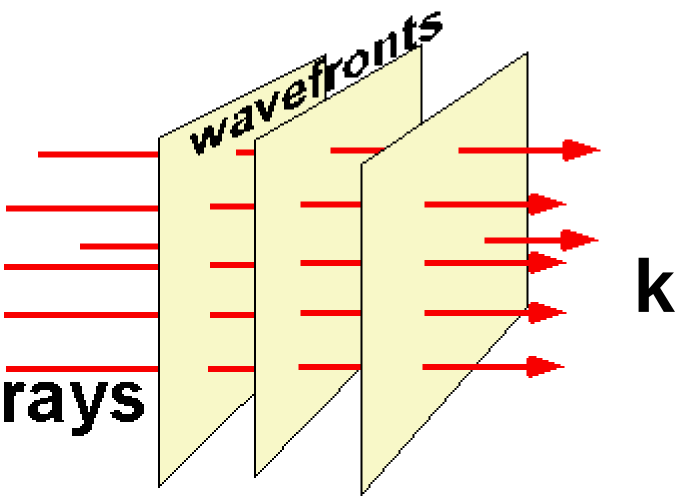
For a spherical wave e^{ikr} the phase-fronts are circles, see next page.

Note: A wavefront will be represented by a whole bundle of rays

Wavefronts and Rays

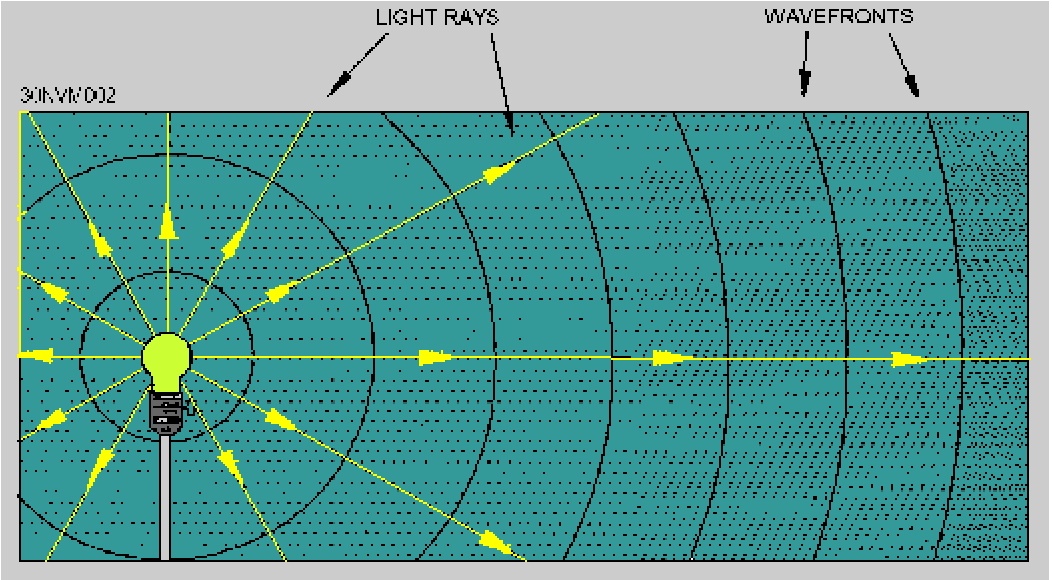
Harmonic plane-waves

- Flat wavefronts
- Rays parallel to **k**



Spherical waves

- Spherical wavefronts
- Outward rays



Poynting vector (Hecht 3.3)

From Phys. 241 you should have seen that the EM energy density is (SI units)

$$u = u_E + u_B$$

$$= \frac{\epsilon}{2} \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B}$$

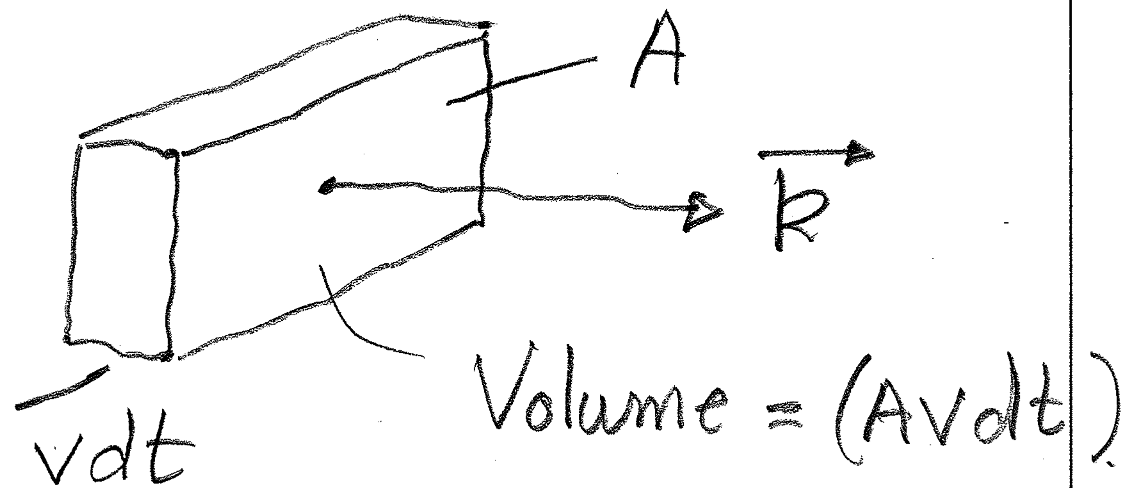
$$= \frac{\epsilon}{2} E^2 + \frac{1}{2\mu_0} B^2$$

But we have seen for a harmonic soln. $E = vB$, so using $v^2 = c^2/n^2 = 1/(\epsilon_0 \mu_0 n^2)$

$$u = \epsilon E^2, \quad \epsilon = n^2 \epsilon_0 .$$

If this represents a harmonic EM wave propagating at velocity $v = c/n$ then energy will be propagating:

How much?



In a short time dt an energy (Avdt)ϵE^2 crosses the area A, giving the energy/area/time S

$$S = \frac{(Avdt)\epsilon E^2}{Adt} = \epsilon_0 n c E^2$$

S - units $J L^{-2} T^{-1} \longrightarrow W/m^2$.

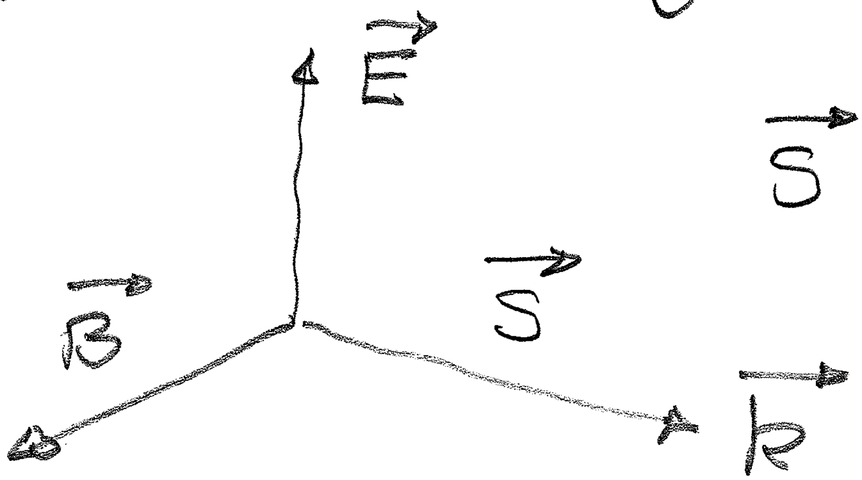
$$\boxed{S = \epsilon_0 n c E^2 = \epsilon_0 n c \vec{E} \cdot \vec{E}}$$

The magnitude of the energy flow varies like E^2.

How about the direction of energy flow?
Incorporated in the <u>Poynting vector</u>

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

For our harmonic plane-wave soln.
$\vec{E}$, $\vec{B}$ & $\vec{k}$ form a right-handed set



$\vec{S} \parallel \vec{k}$

=> energy flow in direction of $\vec{k}$

$$|\vec{S}| = S = \frac{1}{\mu_0} |\vec{E}||\vec{B}|$$

$$= \frac{1}{\mu_0} EB = \frac{1}{\mu_0 v} E^2$$

$$= \epsilon_0 n c E^2 \quad \text{QED.}$$

This defenition of the Poynting vector gives the correct direction of energy flow and magnitude.

Time-averaged energy flow

More carefully the magnitude of energy flow is.

$$S(\vec{r},t) = \epsilon_0 n c \vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r},t)$$

it depends on $\vec{r}$ & t. Let's use our harmonic plane-wave (real)

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

giving

$$= |\vec{E}_0|^2$$

$$S(\vec{r},t) = \epsilon_0 n c E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

Now

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

So

$$S(\vec{r},t) = \frac{1}{2}\epsilon_0 n c E_0^2$$

$$+ \frac{1}{2}\epsilon_0 n c E_0^2 \cos(2(\vec{k}\cdot\vec{r} - \omega t + \varepsilon))$$

The second term oscillates rapidly in space & time, on a timescale of half an optical period – detectors will average over a time much larger, so we neglect oscillating term

$$\boxed{\langle S \rangle_T = I = \frac{1}{2}\epsilon_0 n c E_0^2}$$

I – irradiance in W/m^2 (SI)

E_0 – peak electric field in V/m.

I is often called the intensity or irradiance

Photons (Hecht 3.3.3)

According to the classical theory of the EM field the energy can assume any value,

$$I = \frac{1}{2}\epsilon_0 n c |\vec{E}_0|^2$$

However, according the quantum theory of the EM field, Quantum Electro-dynamics (QED), the field energy comes in discrete quanta or photons. The energy of a single photon is

$$\boxed{E = \hbar\omega = h\nu}, \quad \hbar = h/2\pi$$

with $\hbar = 10^{-34}$ Js is Planck's const.

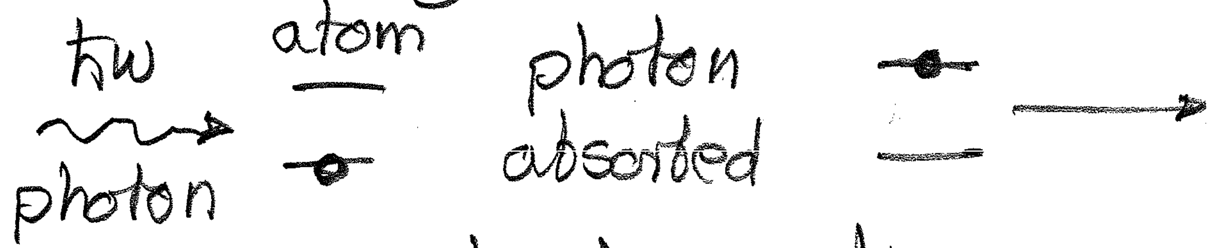
For a typical laser the number of photons involved is so large we are not aware that the number of photons is discrete.

The area of <u>Quantum Optics</u> deals with those cases where the quantized nature of the optical field is relevant, often in cases with a low number of photons.

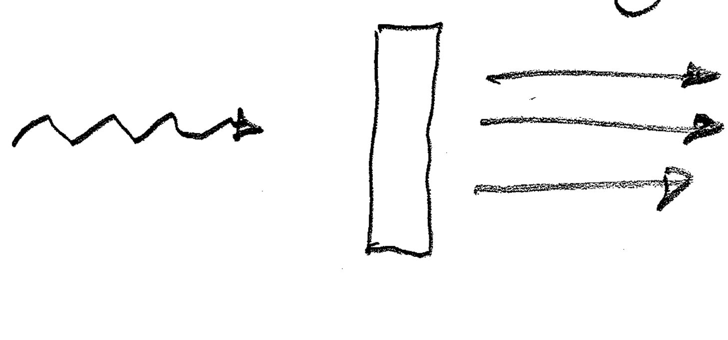
Quantum theory also associates a momentum with each photon

$$\boxed{\vec{p} = \hbar \vec{k}}$$

This implies a light field can push a material object



absorbing medium



Radiation pressure.