Three-dimensional wave equation (Hecht 2.7-9)

The 1D wave equation for a scalar field $\psi(x,t)$ is

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

What is 3D form for $\psi(\vec{r},t) = \psi(x,y,z,t)$?

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$= \nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}.$$

It is a linear PDE for ψ, often written in standard form as.

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{r},t) = 0$$

Laplacian operator.

<u>3D Harmonic waves</u>: These take the form

$$\psi(x,y,z,t) = A\sin(k_x x + k_y y + k_z z - \omega t + \varepsilon)$$

Substituting in the wave equation

$$\nabla^2\psi = -(k_x^2 + k_y^2 + k_z^2)\psi = -k^2\psi$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi$$

leading to the dispersion relation

$$\left[-(k_x^2 + k_y^2 + k_z^2) + \frac{\omega^2}{v^2}\right]\psi = 0$$

or $\omega^2 = v^2(k_x^2 + k_y^2 + k_z^2)$.

like magnitude squared $|\vec{k}|^2 = k^2$ of a vector

$$\vec{k} = \hat{i}k_x + \hat{j}k_y + \hat{z}k_z$$

then $\omega = vk$ as before, with k the propagation number of the 3D wave

$\vec{k}$ is called the <u>propagation vector</u>, and for a monochromatic field of angular temporal frequency $k = \omega/v$.

To proceed consider

$$k_x x + k_y y + k_z z = \vec{k} \cdot \vec{r}$$

with $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ the position vector.

So the harmonic solution may be written as

$$\psi(x,y,z,t) = A\sin(k_x x + k_y y + k_z z - \omega t + \varepsilon)$$

$$= \psi(\vec{r},t) = A\sin(\vec{k}\cdot\vec{r} - \omega t + \varepsilon).$$

Q: What direction is this harmonic wave propagating in, and what does the wave look like $(t=0)$?

$$\psi(\vec{r}, t=0) = A\sin(\phi(\vec{r}, t=0))$$

$$\text{phase} = \phi(\vec{r}, t=0) = \vec{k}\cdot\vec{r} + \varepsilon.$$

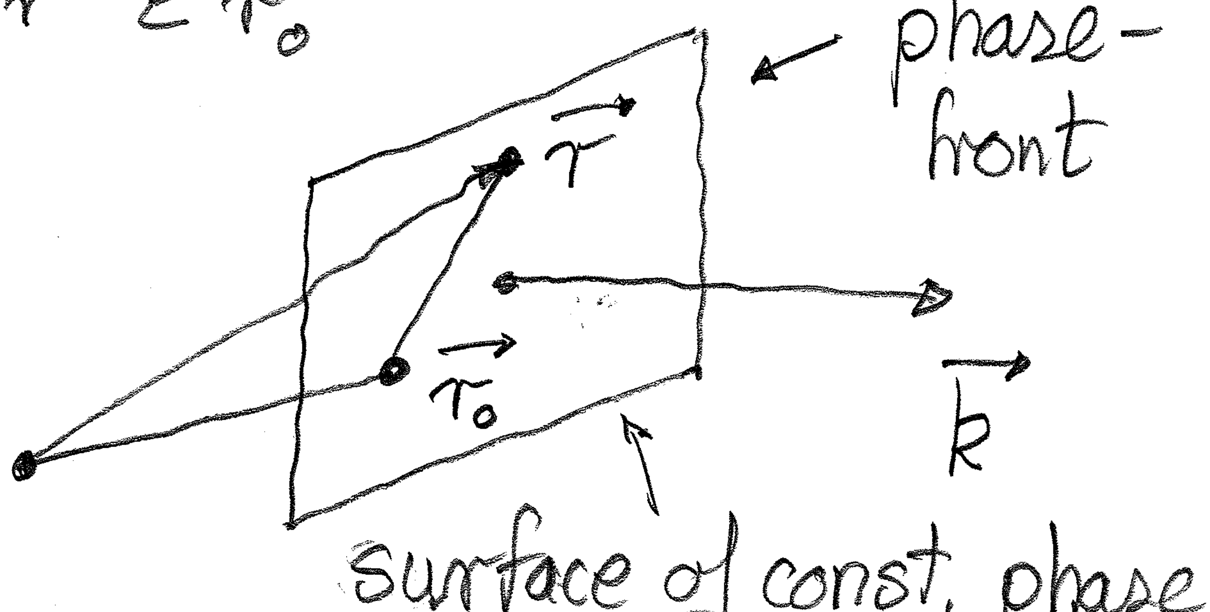
To answer this set a reference point $\vec{r}_0 = \hat{i}x_0 + \hat{j}y_0 + \hat{k}z_0$ in space

$$\varphi(\vec{r}_0, t=0) = \vec{k}\cdot\vec{r}_0 + \varepsilon$$
$$\varphi(\vec{r}, t=0) = \vec{k}\cdot\vec{r} + \varepsilon$$

Subtracting these gives the phase diff.

$$\Delta\varphi(\vec{r}, \vec{r}_0) = \varphi(\vec{r}, t=0) - \varphi(\vec{r}_0, t=0)$$
$$= \vec{k}\cdot(\vec{r} - \vec{r}_0)$$

Thus, if $(\vec{r} - \vec{r}_0) \perp \vec{k}$, $\Delta\varphi = \vec{k}\cdot(\vec{r} - \vec{r}_0) = 0$, and the phase is the same $\vec{r}$ & $\vec{r}_0$



If $\vec{r}$ & $\vec{r}_0$ lie in the same plane $\perp$ to $\vec{k}$ they have the same phase $\varphi' = \varphi(\vec{r}_0, t=0)$, hence the same wave amplitude $\psi = A \sin \varphi'$.

- Harmonic waves have uniform field profile transverse or $\perp$ to $\vec{k}$, they are called <u>plane-waves</u>.

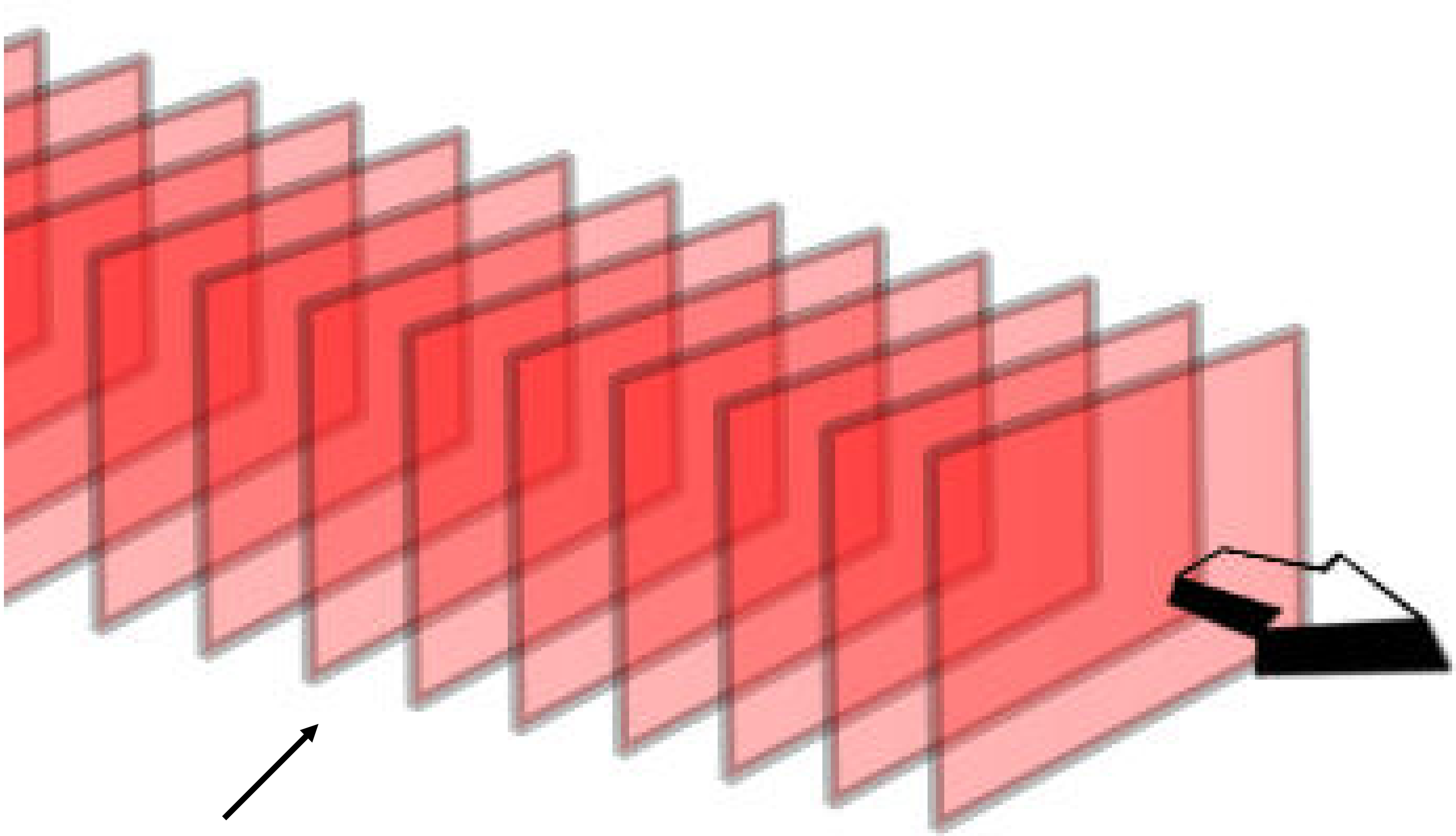
<u>Example</u>: take $\vec{k} = k\hat{i}$, then

$$\psi(\vec{r}, t) = A \sin(\vec{k} \cdot \vec{r} - \omega t + \varepsilon)$$
$$= A \sin(kx - \omega t + \varepsilon)$$

which reduces to our 1D harmonic wave propating along x-axis!

[General: the harmonic wave or plane-wave propagates along the direction of $\vec{k}$.]

Plane-wave fronts propagating perpendicular to the propagation wavevector **k**



Surfaces of constant phase

Direction cosines: These relate to a different form for the phase ($\varepsilon = 0$)

$$\varphi(\vec{r}, t=0) = \vec{k} \cdot \vec{r}$$

$$= \vec{k} \cdot (\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= (\vec{k} \cdot \hat{i})x + (\vec{k} \cdot \hat{j})y + (\vec{k} \cdot \hat{k})z$$

Now $(\vec{k} \cdot \hat{i}) = |\vec{k}|\,|\hat{i}| \cos\theta_x$, θ_x angle between $\vec{k}$ and x-axis. We write

$$\left.\begin{aligned} (\vec{k} \cdot \hat{i}) &= k\alpha, \quad \alpha = \cos\theta_x \\ (\vec{k} \cdot \hat{j}) &= k\beta, \quad \beta = \cos\theta_y \\ (\vec{k} \cdot \hat{k}) &= k\gamma, \quad \gamma = \cos\theta_z \end{aligned}\right\} \text{direction cosines}$$

$$\varphi(\vec{r}, t=0) = k(\alpha x + \beta y + \gamma z),$$

$$\vec{k} = k(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$$

$$\Longrightarrow \alpha^2 + \beta^2 + \gamma^2 = 1$$

The superposition principle

The 3D wave equation, like the 1D Eq., is linear

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0,$$

so if $\psi_{1,2}$ both obey the 3D Eq. then any linear combination of them is also a solution of the 3D wave eqn.

$$\psi(\vec{r},t) = a\psi_1(\vec{r},t) + b\psi_2(\vec{r},t).$$

Consider superposing two harmonic or plane-waves of equal ω but with different propagation directions.

$$\psi(\vec{r},t) = A_1\sin(\vec{k}_1\cdot\vec{r} - \omega t + \varepsilon_1) + A_2\sin(\vec{k}_2\cdot\vec{r} - \omega t + \varepsilon_2)$$

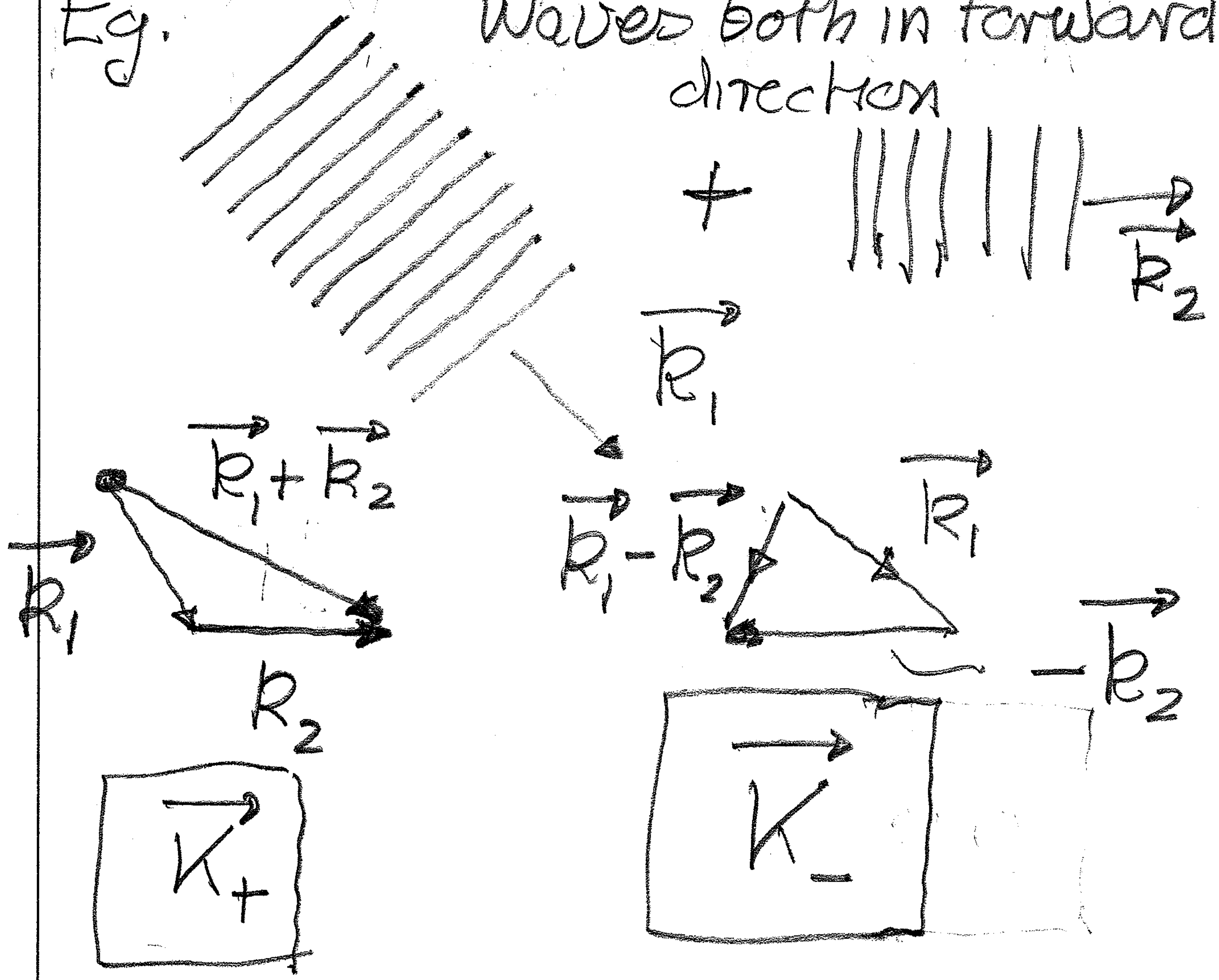
$|\vec{k}_1| = k_1 = k_2 = \omega/v = k$. As in 1D case this produces <u>interference</u>. For simplicity set $\mathcal{E}_1 = \mathcal{E}_2 = 0$, $A_1 = A_2 = A$, then

$$\psi(\vec{r}, t) = 2\sin(\vec{K}_+ \cdot \vec{r} - \omega t)\cos(\vec{K}_- \cdot \vec{r})$$

with

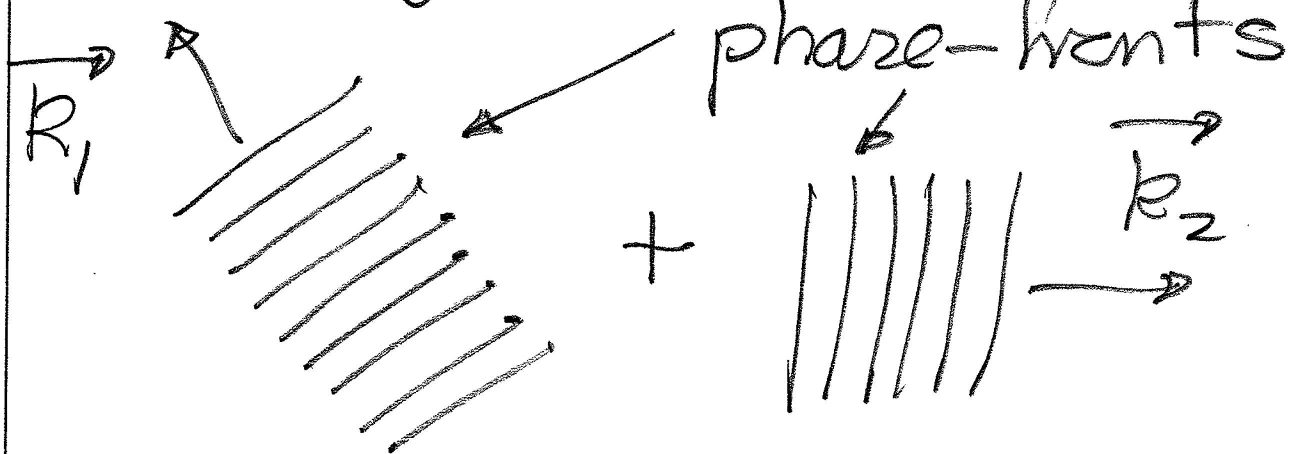
$$\vec{K}_+ = \frac{1}{2}(\vec{k}_1 + \vec{k}_2), \quad \vec{K}_- = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$$

Eg. Waves both in forward direction



See PW_same.avi for illustration.
Explain axes, color coding etc.
Point to the 2 components of the interference pattern assoc. with $\vec{K}_+$ and $\vec{K}_-$, @ moving fringes with time...

Challenge yourself with the case



in PW_opposite.avi...

By adding several plane-waves with different $\vec{k}_j$ we obtain very complicated interference patterns

Conversely, complicated wave fields can be synthesized into plane-wave component. Basis for Fourier analysis & transforms (Opti 280) & Fourier optics (Opti 330).

Complex representation

Consider a 3D plane-wave

$$\Psi(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

We often express this in complex form

$$\Psi(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)}$$

with the tacit assumption that we always take the real part to get the physical field.

The complex plane-wave is of ultility as it's derivatives are simple:

$$\left.\begin{aligned} \nabla\psi &= i\vec{k}\psi, & \nabla &\equiv i\vec{k} \\ \nabla^2\psi &= -k^2\psi & \nabla^2 &\equiv -k^2 \\ \frac{\partial\psi}{\partial t} &= -i\omega\psi, & \frac{\partial}{\partial t} &\equiv -i\omega \\ \frac{\partial^2\psi}{\partial t^2} &= -\omega^2\psi, & \frac{\partial^2}{\partial t^2} &\equiv -\omega^2 \end{aligned}\right\}$$

We shall use the equivalances for the complex representation of a plane-wave with $\vec{k}$ & ω. — they do not apply to the real solution, eg. ($@ \; t=0$)

$$\nabla\cos(\vec{k}\cdot\vec{r}) \neq \vec{k}\cos(\vec{k}\cdot\vec{r})$$

Eg. Take the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

For the complex plane-wave
$\nabla^2\psi \equiv -k^2\psi$, $\partial^2\psi/\partial t^2 \equiv -\omega^2\psi$, so

$$-k^2\psi = -\frac{\omega^2}{v^2}\psi,$$

$$\Rightarrow k^2 = \omega^2/v^2, \quad k = \omega/v.$$

<u>Spherical waves</u>

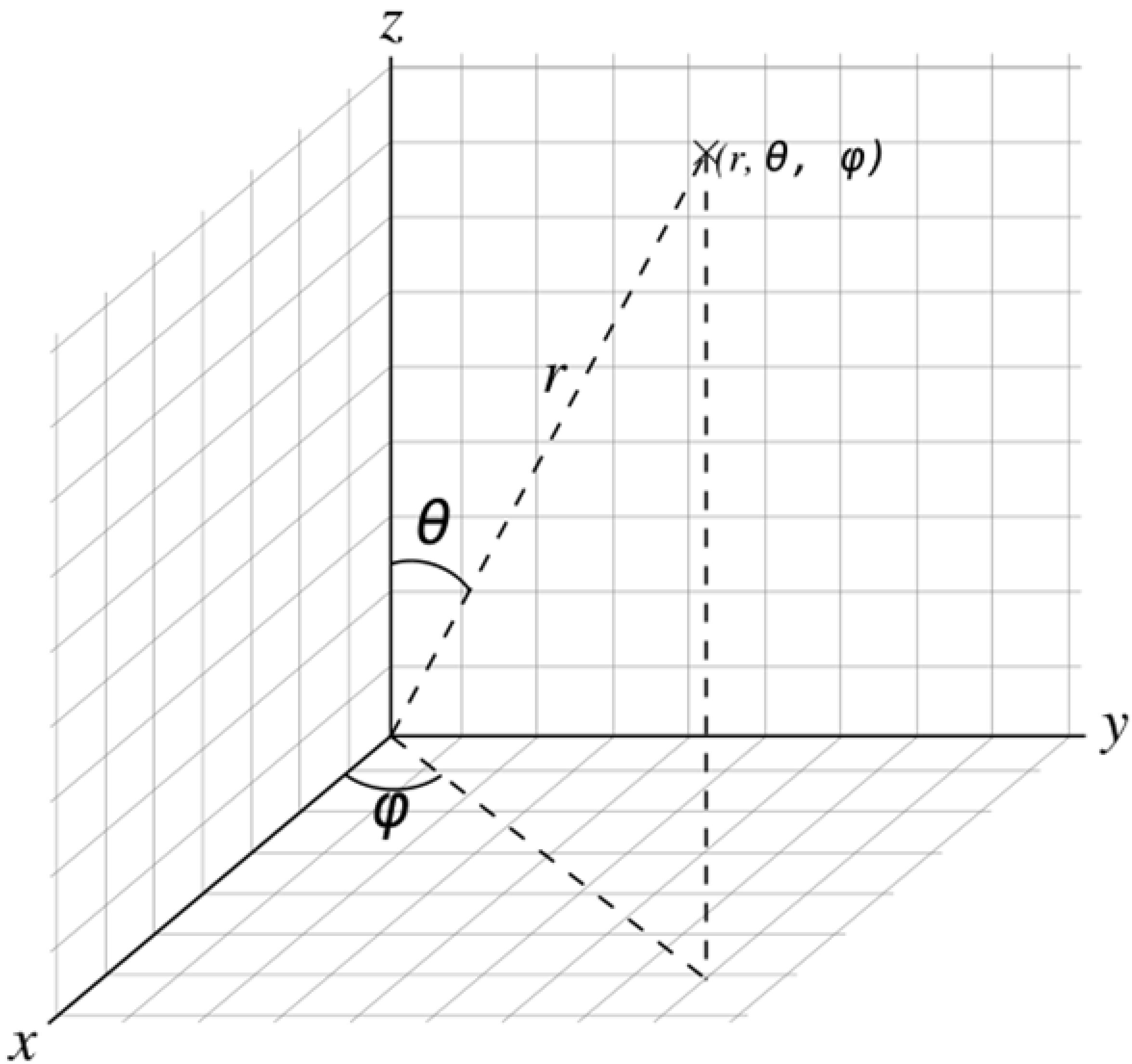
So far we have considered plane-waves in a Cartesian coordinate system giving rise to flat phase-fronts - see figure. We can also consider spherical waves using spherical coordinates (r, θ, ϕ). Here I will quote the result for the spherical wave without the math (see Hecht 2.9). For a spherically symmetric wave wrt the origin

$$\psi(r, \theta, \phi, t) = \frac{\mathcal{A}}{r} \cos\left(kr \mp \omega t + \varepsilon\right)$$

$$= \psi(r, t) \qquad \text{no vectors}$$

- Harmonic spherical wave, independent of θ, ϕ
- $k = \omega / c$ as before
- $(-)$ wave outward propagating
  $(+)$ " inward " "
- ε initial phase
- r distance from origin
- $\mathcal{A}$ – source strength.

Cartesian and Spherical coordinate systems



- $1/r$ dependence means the wave decays away from the origin (source), think of a water wave on a pond after dropping a pebble (2.24)
- Surfaces of constant phase are $kr = \text{const.}$, so they are spheres (circles in a plane, eg. $z=0$)
- Outward or inward propating spherical wavefronts (see Hecht Fig. 2.25), <u>Spherical.avi</u>

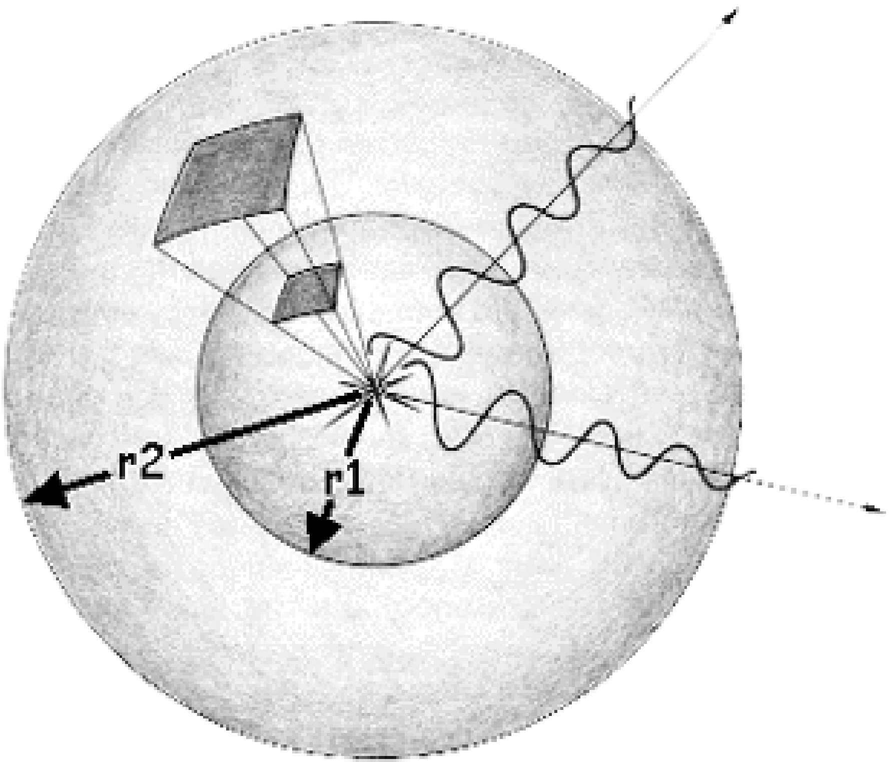
<u>Complex representation</u>

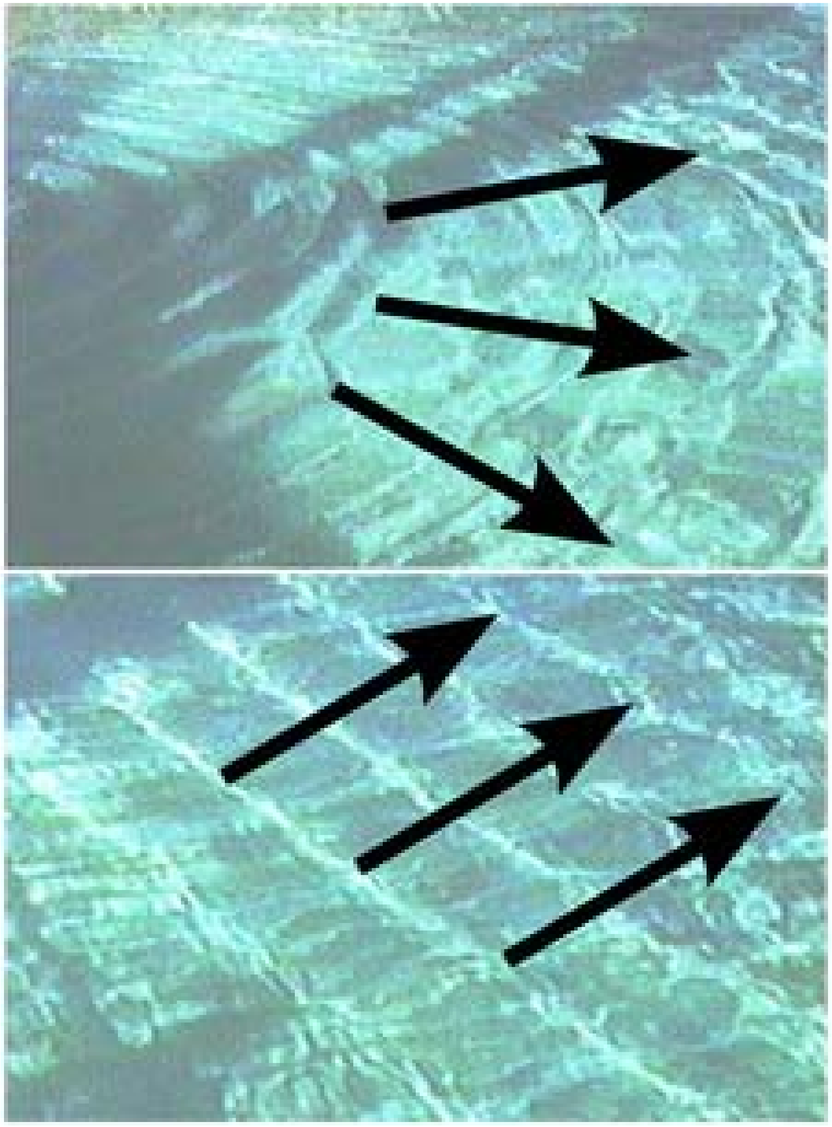
Consider a harmonic spherical wave centered on $\vec{r}_0$ ($\varepsilon = 0$)

$$\psi(\vec{r},t) = \frac{\mathcal{A}}{|\vec{r}-\vec{r}_0|}\cos(k|\vec{r}-\vec{r}_0| - \omega t)$$

Plane and cylindrical water waves

Spherical wave





The complex representation of this is

$$\psi(\vec{r},t) = \frac{A}{|\vec{r}-\vec{r}_0|} e^{i(k|\vec{r}-\vec{r}_0|-\omega t)}$$

If $\vec{r}_0 = \vec{0}$, this reduces to

$$\psi(r,t) = \frac{A}{r} e^{i(kr-\omega t)}$$

Summary.

You have now seen the basic physics of waves in 1D & 3D, i.e wave eqn, harmonic waves, frequency, wavelength...
In coming lectures we shall see how EM fields may be described as transverse EM waves, and we shall use the solutions from this section.

The wave equation in spherical polar coordinates is

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(r,\theta,\phi,t) = 0$$

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Complicated, but if $\psi(r,\theta,\phi,t) = \psi(r,t)$ is spherically symmetric, independent of θ & ϕ, $\partial\psi/\partial\theta = \partial\psi/\partial\phi = 0$, and

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

So

$$\nabla^2\psi(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)$$

$$= \frac{1}{r^2}\left[r^2\frac{\partial^2\psi}{\partial r^2} + 2r\frac{\partial\psi}{\partial r}\right]$$

$$= \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).$$

The 3D wave equation becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

multiply by r to get

$$\frac{\partial^2}{\partial r^2}(r\psi) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi) = 0$$

or

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)(r\psi(r,t)) = 0$$

Just like 1D wave eqn. for $(r\psi(r,t))$

So harmonic solution

$$(r\psi(r,t)) = \mathcal{A}\cos(kr \mp \omega t + \varepsilon)$$

or

$$\psi(r,t) = \frac{\mathcal{A}}{r}\cos(kr \mp \omega t + \varepsilon)$$