One-dimensional waves (Hecht 2.1–2.6)

Basic physics of waves in 1D. Classification

- Longitudinal waves – wave displaced in direction of propagation
- Transverse waves – wave displacement in plane transverse or perpendicular to direction of propagation

See slinky movie & Hecht Fig. 2.1.

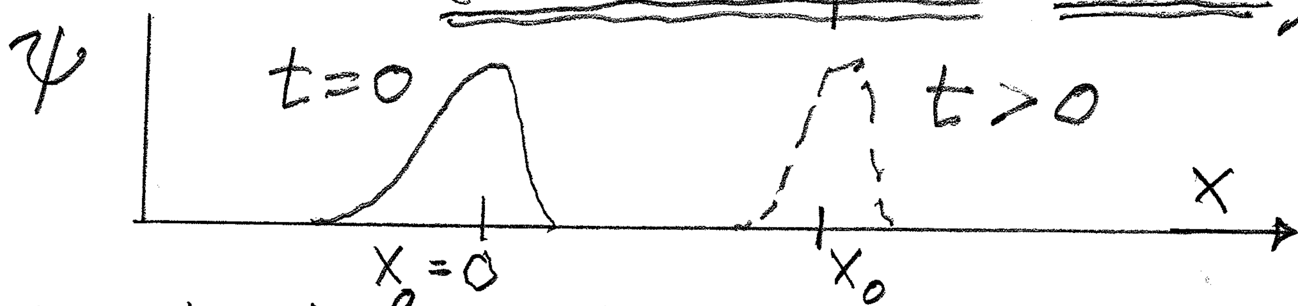
Physical examples

- Longitudinal pressure waves in media are sound waves – waves of compression – see web links.
- Waves on the surface of water, water surface is displaced transverse to the propagation direction – see web links.

- Light waves – light is a transverse wave phenomenon, EM fields oscillate transverse to direction of propagation. – more later

1D Travelling waves

Consider a moving displacement representing a transverse wave, eg. a string (Hech Fig. 2.2), or a water wave (see examples & web link)

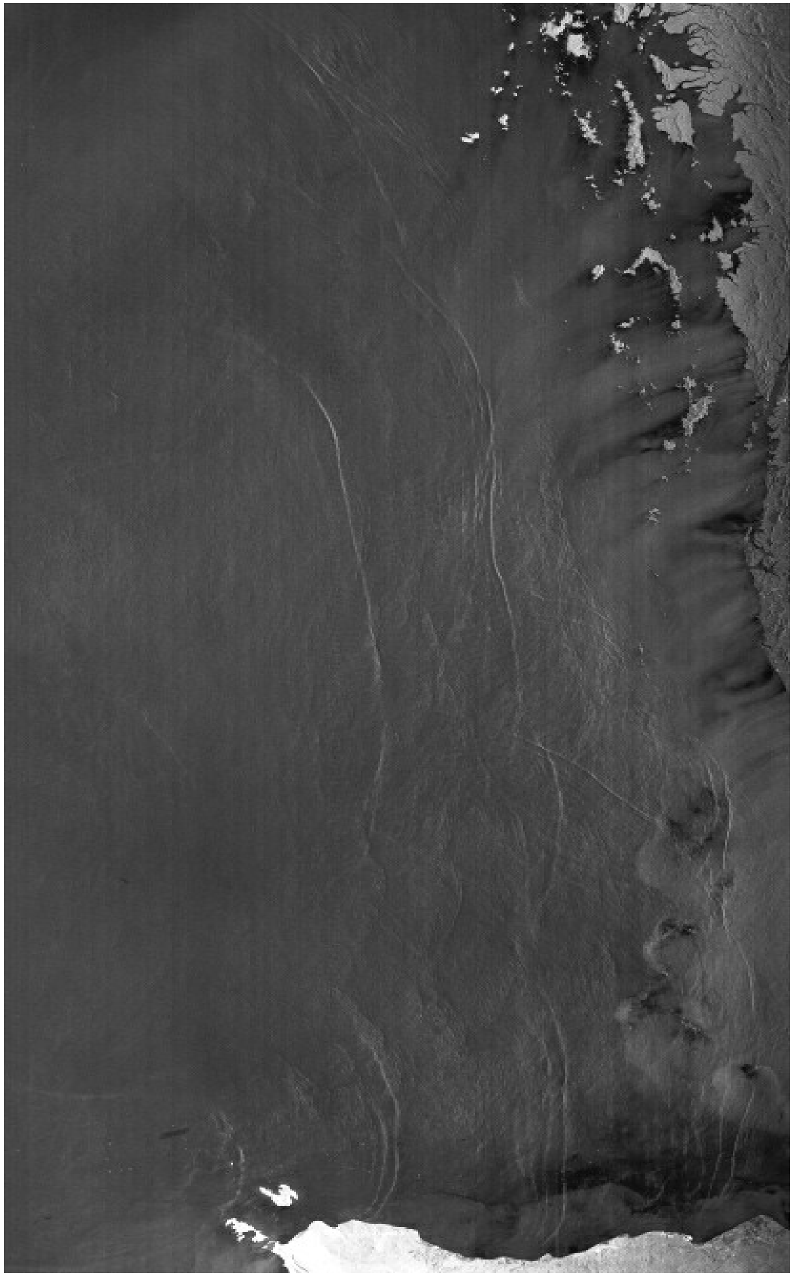


The displacement is ψ – water or string height

$$\psi \equiv \psi(x,t)$$

$$f(x) = \psi(x,t)\Big|_{t=0}$$

initial displacement @ $t=0$



1D water waves

- Ocean waves
- Elevated water bump in a canal

As time advances assume it's profile $f(x)$ stays the same but it's peak moves at velocity v - property of waves

$$\boxed{\psi(x,t) = f(x - vt)}$$

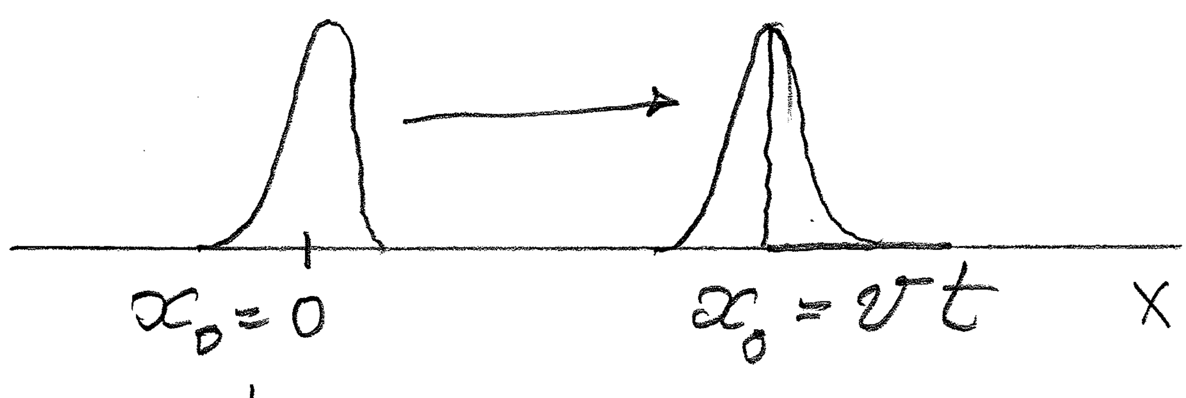
v - wave velocity (sound, light, ...)

Eg. $f(x) = e^{-ax^2}$, $a > 0$, so

$$\psi(x,t) = e^{-a(x-vt)^2} \quad \text{(Gaussian)}$$
$$= e^{-a(x-x_0(t))^2}, \quad x_0(t) = vt$$

At $t = 0$, peak @ $x_0(0) = 0$

At $t > 0$, peak @ $x_0(t) = vt$



See Fig. 2.4 Hecht

this form

$$\psi(x,t) = f(x \mp vt)$$

is characteristic of travelling waves of velocity v, the (-) correspond to a wave propagating along the $+x$ axis, the (+) propagation along the $-x$ axis.

<u>One-dimensional wave equation</u>

Goal: find an equation of motion for ψ

$$\psi(x,t) = f(x \mp vt) = f(x')$$

with $x' = x \mp vt$. Consider partial diffs.

$$\frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x'}\left(\frac{\partial x'}{\partial x}\right) = \frac{\partial f}{\partial x'}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial x'}\left(\frac{\partial x'}{\partial t}\right) = \mp v \frac{\partial f}{\partial x'} = \mp v \frac{\partial \psi}{\partial x}$$

(used the <u>chain rule</u>). This yields the partial differential equation for ψ

$$\frac{\partial \psi}{\partial t} = \mp v \frac{\partial \psi}{\partial x}$$

→ depends on direction ($\pm$) of propagation

Take second partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\right) = \left[\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\right)\right]\left(\frac{\partial x'}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\mp v \frac{\partial f}{\partial x'}\right) = \left[\frac{\partial}{\partial x'}\left(\mp v \frac{\partial f}{\partial x'}\right)\right]\left(\frac{\partial x'}{\partial t}\right)$$

$$= \mp v \frac{\partial^2 f}{\partial x'^2} \cdot (\mp v)$$

$$= v^2 \frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 \psi}{\partial x^2}$$

Thus we obtain

$$v^2 \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial t^2}$$

giving the 1D wave equation

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}}$$

It is a partial differential equation (PDE) for the scalar wave $\psi(x,t)$.

We shall ultimately find a wave equation for the propagation of transverse EM fields.

Physical examples

- Water waves
- Sound waves - acoustics
- Radio waves - EM

⋮

Harmonic waves

In the homework you shall verify that a specific solution of the wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Harmonic wave

$$\psi(x,t) = A\sin(kx \mp \omega t + \varepsilon)$$

A - wave amplitude

k - propagation number

ω - angular temporal frequency

ε - initial phase @ $(x=0, t=0)$

∓ - determines direction of propagation along the x-axis.

You will also find the relation

$$\boxed{\omega = kv}$$

⟶ dispersion relation

Here we consider the particular case of propagation along the $+x$ axis $(-)$ and $\varepsilon = 0$, so

$$\psi(x,t) = A\sin(kx - \omega t)$$

and thus the initial condition $(t=0)$ is

$$\psi(x,t)\big|_{t=0} = A\sin(kx).$$

This waveform is periodic is space with period λ called the <u>wavelength</u>

$$k(x+\lambda) = kx + 2\pi$$

$$\Rightarrow \boxed{k = \frac{2\pi}{\lambda} = 2\pi\kappa}$$

wave number or <u>spatial frequency</u> $\kappa = 1/\lambda$

We have

$$kx = 2\pi\left(\frac{x}{\lambda}\right) = 2\pi\kappa x \rightarrow \text{angle}.$$

which we need dimensionally.

The initial waveform is periodic in space with period λ, hence the name harmonic.
We can also write

$$\psi(x,t) = A\sin\left(k\underbrace{\left(x-\left(\tfrac{\omega}{k}\right)t\right)}_{x'}\right)$$
$$= f(x-vt)$$

with $\boxed{v = \omega/k}$ the velocity, or

$$\omega = vk = \frac{2\pi v}{\lambda} \quad \text{— angular temporal frequency}$$

$$= 2\pi\nu \quad \text{— } \nu \text{ (temporal frequency)}$$

giving

$$\boxed{\nu = v/\lambda}$$

$\boxed{\nu - nu}$. Returning to our solution

$$\psi(x,t) = A\sin(kx-\omega t)$$
$$= A\sin(kx - 2\pi\nu t).$$

For a given spatial position x, the wave is periodic in time t with period τ such that

$$\omega\tau = 2\pi\nu\tau = 2\pi$$

or

$$\boxed{\tau = \frac{1}{\nu} = \frac{2\pi}{\omega} = \frac{2\pi}{kv} = \frac{\lambda}{v} = \frac{1}{\bar{k}v}}$$

hence the name harmonic wave.
Harmonic waves are also called monochromtic wave since they have a single frequency ν or angular frequency $\omega = 2\pi\nu$. Here they also have a single spatial frequency $\bar{k} = 1/\lambda = \frac{k}{2\pi}$.

Consider the solution with initial phase

$$\psi(x,t) = A\sin(kx - \omega t + \varepsilon)$$

$$= A\cos(\varepsilon)\sin(kx - \omega t)$$

$$+ A\sin(\varepsilon)\cos(kx - \omega t)$$

Thus even a monochromatic field at angular frequency ω generally has two components, a sin-component and a cos-component. Both of these are required to describe a general monochromatic signal at ω.

Phase & phase velocity

Consider again the solution

$$\psi(x,t) = A\sin(kx - \omega t + \varepsilon)$$

$$= A\sin(\varphi(x,t))$$

where the phase of the wave is

$$\varphi \equiv \varphi(x,t) = (kx - \omega t + \varepsilon)$$

ε is the initial phase @ $x=0, t=0$.

For a travelling wave as above the wave motion is evident if one traces the motion of a point of constant amplitude on the wave - eg. a peak or a trough - see weblink.

Points on the wave that have the same value of their phase φ' will have the same value $A\sin\varphi' = \psi$

$$\varphi(x,t) = kx - \omega t + \varepsilon = \varphi'$$

$$\varphi(x+dx, t+dt) = k(x+dx) - \omega(t+dt) + \varepsilon$$

Subtracting these gives

$$k\,dx - \omega\,dt = 0$$

or the <u>phase-velocity</u>

$$v = \frac{dx}{dt} = \frac{\omega}{k}$$

the <u>phase velocity</u> tells us how fast positions of constant phase move.

<u>The superposition principle</u>

Consider we have two solutions ψ_1 & ψ_2 of the wave equation

$$\frac{\partial^2 \psi_1}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi_1}{\partial t^2}$$

$$\frac{\partial^2 \psi_2}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi_2}{\partial t^2}$$

then by adding these we see

$$\frac{\partial^2}{\partial x^2}(\psi_1 + \psi_2) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2}(\psi_1 + \psi_2)$$

Thus we can always superpose or add (or subtract) solutions of the wave equation – this is the principle of superposition and follows from the <u>linear</u> nature of the wave equation – if the wave equation was nonlinear

$$\underbrace{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}_{\text{linear}} + \underset{\substack{\uparrow \\ \text{nonlinear}}}{\psi^2}$$

$$(\psi_1+\psi_2)^2 = \psi_1^2 + 2\psi_1\psi_2 + \psi_2^2 \neq \psi_1^2 + \psi_2^2$$

<u>Matlab</u> examples of superposing two or more waves – <u>interference</u>

$$\psi(x, t=0) = \sin(k_1 x) + \sin(k_2 x)$$

$\longrightarrow$ <u>carrier</u> fringes + <u>envelope</u>

One obtains interference or beats distrubution as the components go in and out of phase .. see simulations, also homework. (Interference.m)

Can turn this around: synthesize more general waveforms by superposing sin & cos solutions. This is the basis of Fourier series.

Example of superposition in space & time $\boxed{\text{Interference.m}}$

$$\psi(x,t) = \sin(k_1(x-v_1 t)) + \sin(k_2(x-v_2 t)) + \dots$$

each term satisfies the wave equation. Matlab simulation

→ phase velocity & group velocity, envelope & carrier

Complex representation

So far we have tacitly assumed the scalar field $\psi(x,t)$ describing the wave is real. This makes sense since physical fields are real. However, there is a very usefull complex representation of harmonic waves based. Recall that (Hecht 2.5)

$$\cos\theta = \frac{(e^{i\theta} + c.c)}{2}, \quad \sin\theta = \frac{(e^{i\theta} - c.c)}{2i}$$

where c.c. means complex conjugate, and by the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta = (e^{-i\theta})^*$$

Then consider a real field $\psi(x,t)$

$$\psi(x,t) = A\cos(kx - \omega t + \varepsilon),$$

then we have

$$\Psi(x,t) = A\,\frac{\left(e^{i(kx-\omega t+\varepsilon)} + e^{-i(kx-\omega t+\varepsilon)}\right)}{2}$$

$$= \frac{A}{2}e^{i(kx-\omega t+\varepsilon)} + c.c$$

The part varying as $e^{-i\omega t}$ is often refered to as the "positive" frequency component, $e^{i\omega t}$ as the "negative" frequency component – compare the sin- and cos-components for a harmonic

2 monochromatic real field.

We often use complex representations for harmonic scalar fields

$$\Psi(x,t) = \left(Ae^{i\varepsilon}\right)e^{i(kx-\omega t)}$$

complex amplitude

with the understanding that to obtain the real physical field

we need to take the real part of
the complex field —— complex.

$$\Psi(x,t) = Re\left[A e^{i(kx-\omega t+\varepsilon)}\right]$$

$$= A\cos(kx-\omega t+\varepsilon)$$

real

$$= \frac{1}{2}\left[A e^{i(kx-\omega t+\varepsilon)} + c.c\right]$$