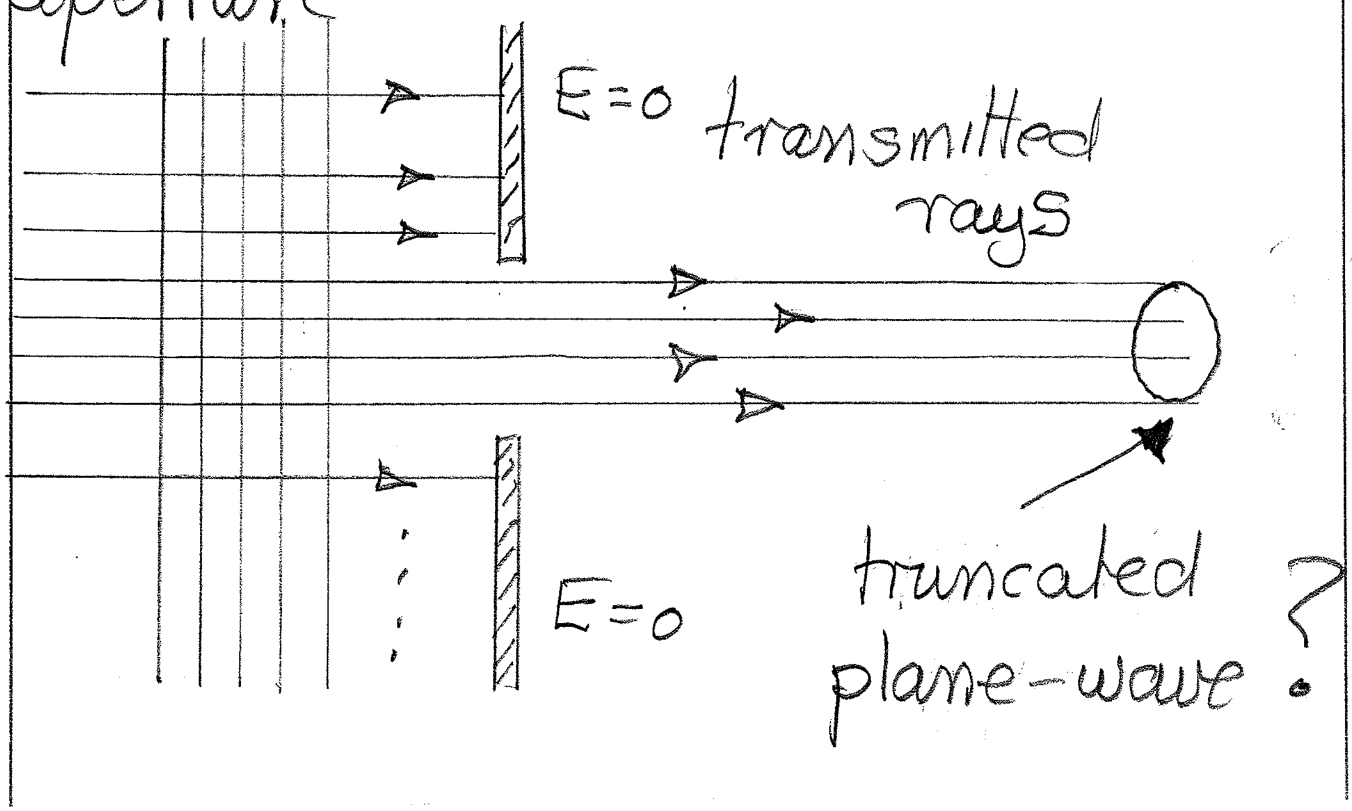
10. Fraunhofer diffraction. (Hecht 10).

In this section we describe the diffraction of electromagnetic waves by planar objects eg. edges or apertures.

Definition (Sommerfeld): "Diffraction is any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction"
— beyond geometric or ray optics, one needs wave optics.

Example: plane-wave falling on an aperture



Notes:

- The aperture truncated the field, E = 0 where field misses aperture.
- Transmitted rays still travel in straight (rectilinear) paths.
- What is the profile of the transmitted field? Uniform over the aperture opening?
- No, much more complicated and interesting, need wave theory.
- At the heart of the debate over whether light is a stream of particles (ray description) or waves. See Spot of Arago at the end.

See Chapter 1 of Hecht

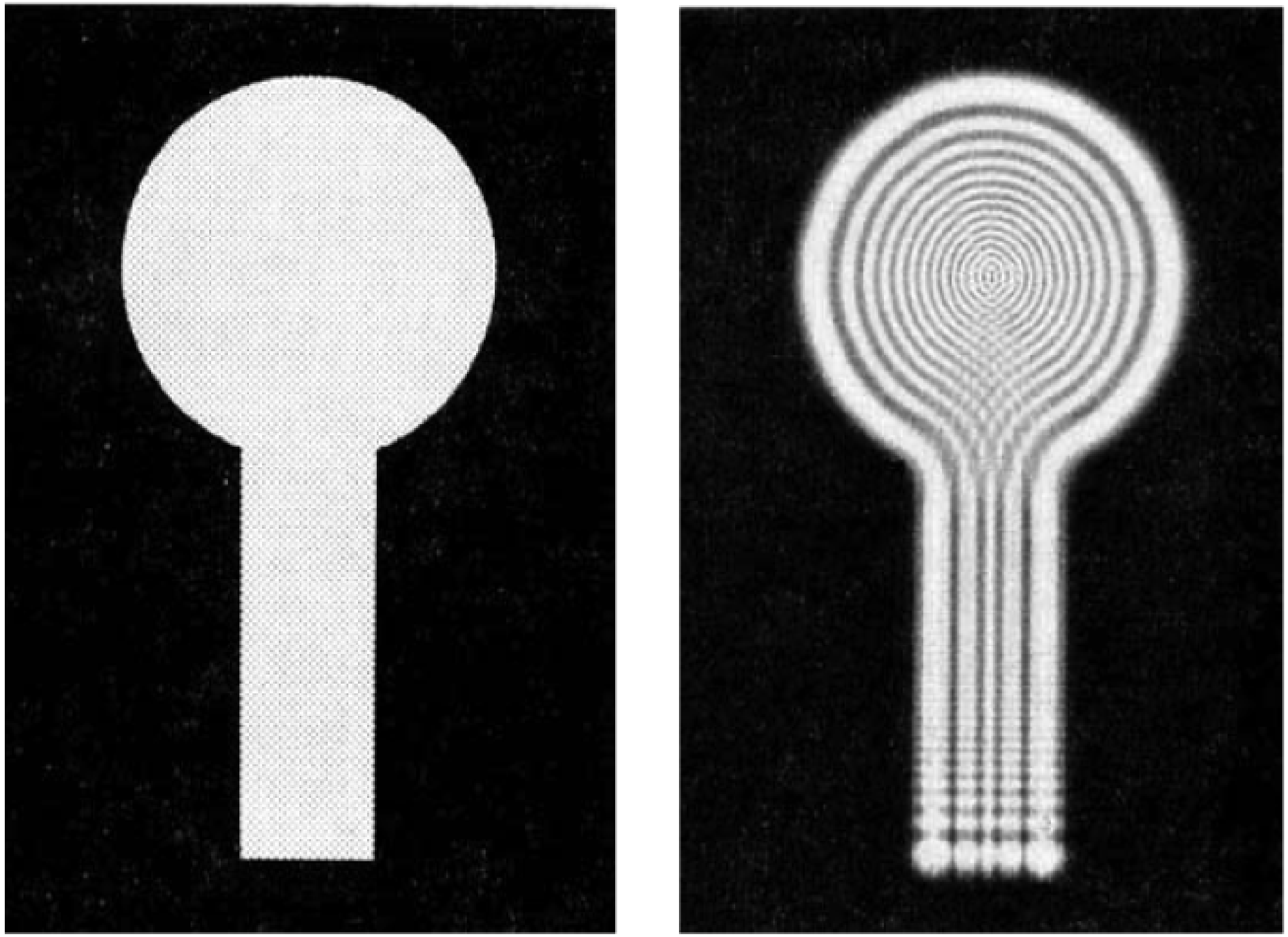


Figure 3-1. Observation of diffraction phenomena
(Reproduced with permission from Reference 1).

Huygens- Fresnel principle

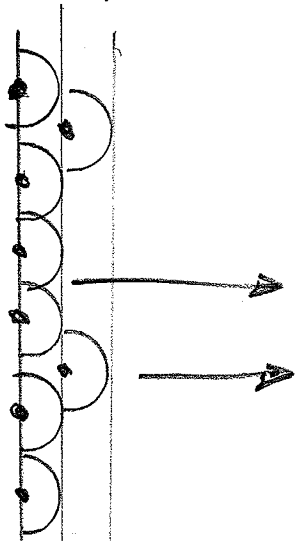
Huygens introduced the wave theory of light in 1678 but it was all but ignored in the 18th century due to the influence of Newton's corpuscular (particle) theory of light. In 1804 Young revived the idea of interference of light, a challenge to Newton's dominance, and by 1818 Fresnel had synthesized the ideas of Huygens & Young into a unified theory of light that could account for observed diffraction effects:

We shall base our discussion on this theory. Ultimately Maxwell's equations & wave equation put diffraction theory on a rigorous basis (1860 – date).

The Huygens-Fresnel principle is a means of calculating how an initial electric field wavefront propagates between planes - It states that each point on a wavefront acts as a secondary source of spherical waves - diffraction is interference between secondary sources!

Example: plane (harmonic) wave

Huygens wavelets

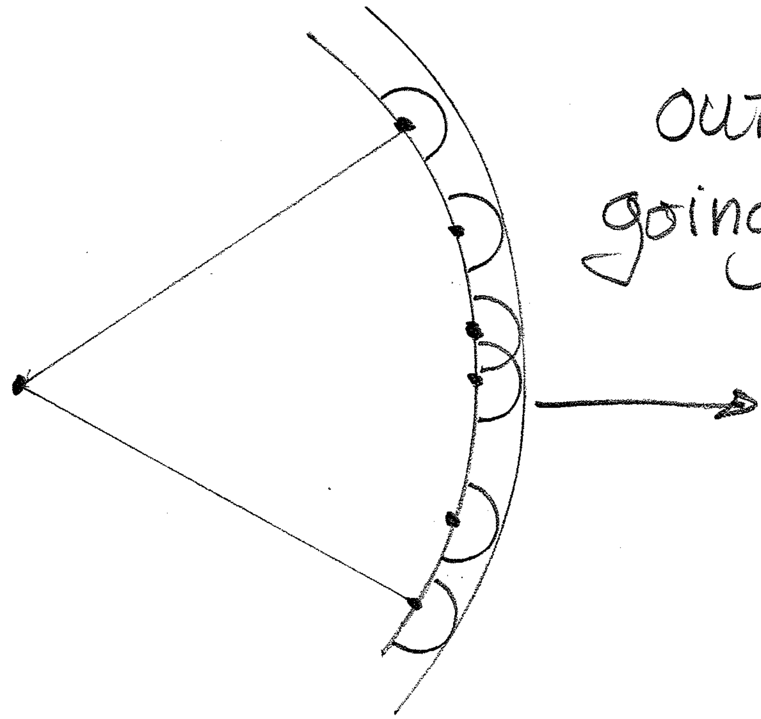


→ Sum secondary spherical waves → propagated plane wave.

Q: Why no backwards propagation?

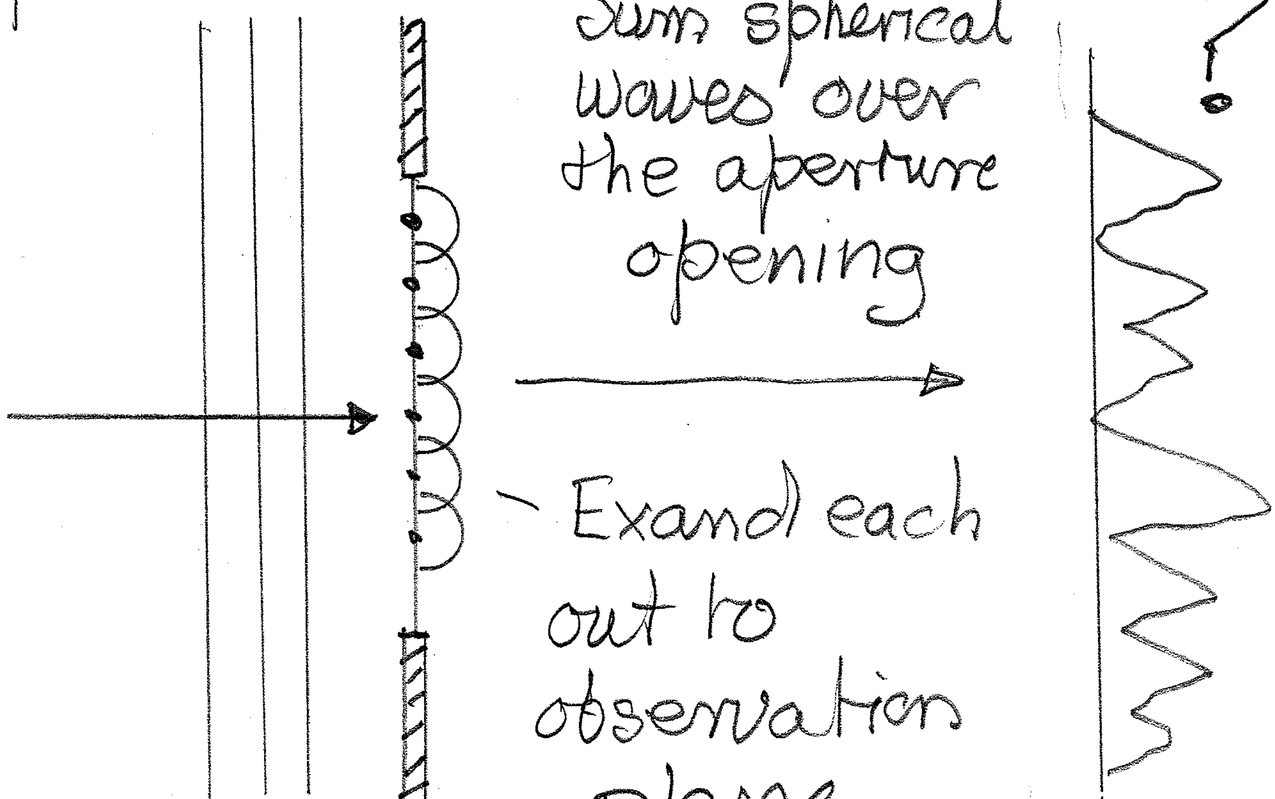
A: this was a problem for Huygen's theory that was resolved later by Fresnel → <u>obliquity factor</u>.

Example: spherical wave



outward going wave.

Example: plane-wave truncated by an aperture.

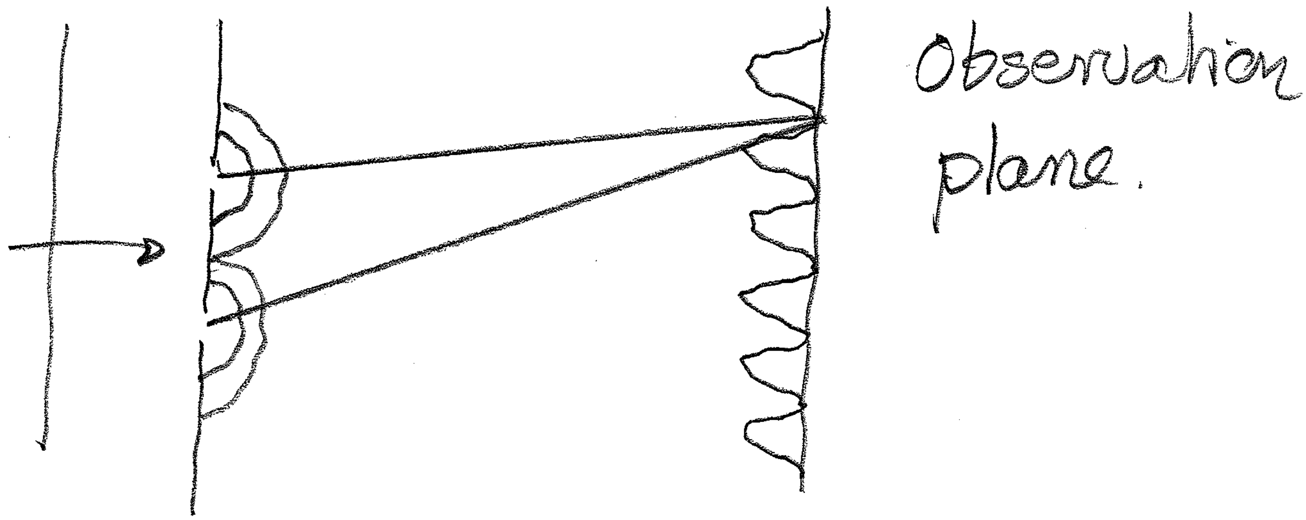


Sum spherical waves over the aperture opening

Exand each out to observation plane.

?

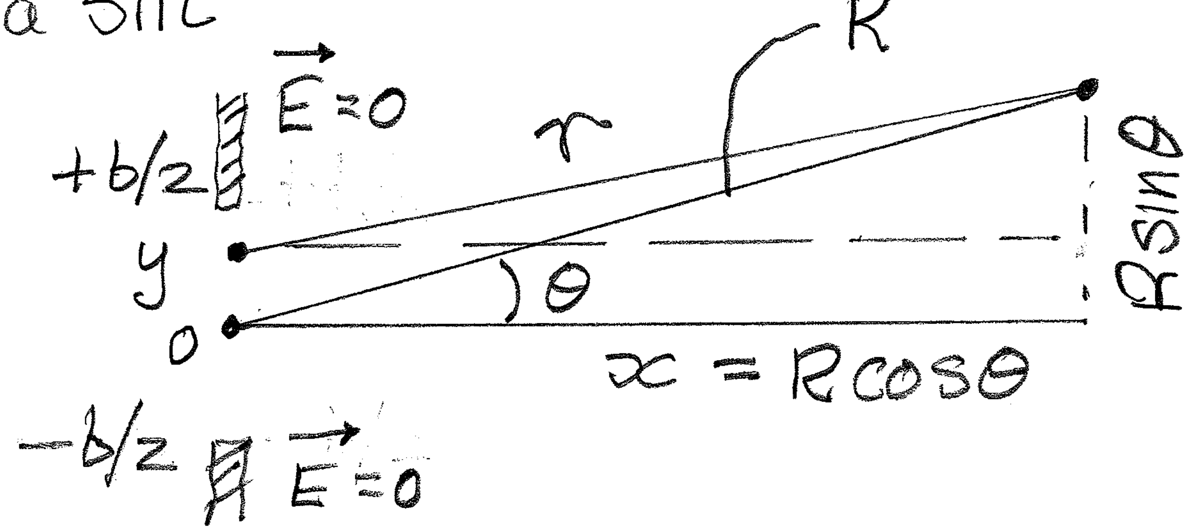
Example: At the end of Sec. 5 on EM waves we already looked at an example of summing spherical waves, namely, Young's two-slit expt.



We know this produces interference. How about a single slit (aperture) of non-zero width 'b'? This corresponds to a sum over a large number of spherical waves emanating from the aperture opening. — Diffraction is ≡ interference between the spherical waves or Huygens wavelets

Fraunhofer vs. Fresnel diffraction

Consider the geometry for diffraction by a slit



From each point labeled $-b/2 \leq y \leq b/2$ on the aperture opening we need to consider an outgoing spherical wave (complex representation)

$$\vec{E}(\vec{r},\omega) = \hat{e}\,\frac{\mathcal{R}}{r}\,e^{ikr}$$

Here we assume $\hat{e}$ is out of the page, and hereafter look at the electric field amplitude. Likewise, $\mathcal{R}$ controls the field amplitude and we set $\mathcal{R}=1$ for simplicity in notation.

Then for each spherical wave

$$E(\vec{r},\omega) \propto \frac{1}{r} e^{ikr}$$

For a given point y on the aperture

$$r^2 = x^2 + (R\sin\theta - y)^2$$
$$= R^2\cos^2\theta + (R^2\sin^2\theta - 2yR\sin\theta + y^2)$$

giving

$$r = \sqrt{R^2 - (2yR\sin\theta - y^2)}$$
$$= R\sqrt{1 - \frac{(2yR\sin\theta - y^2)}{R^2}}$$

We assume $x \gg b, R\sin\theta$, screen far away, eg.

$b \sim 4\mu m, \quad R\sin\theta \sim 100\mu m$

$x = 1cm = 10^4 \mu m.$

$\sin\theta \sim \theta$ for small angles.

Then we may Taylor expand r

$$r = R\sqrt{1 - \frac{(2yR\sin\theta - y^2)}{R^2}}$$

$$\simeq R\left(1 - \frac{1}{2}\frac{(2yR\sin\theta - y^2)}{R^2}\right)$$

$$\simeq R - y\sin\theta + \frac{1}{2}\frac{y^2}{R} + \dots$$

In the Fresnel theory of diffraction we generally have to keep the full form for r, at least the three terms above so we have y & y^2 variation. This produce diffraction effects that vary strongly with distance x.

In the Fraunhofer region we are far enough away that a good approximation is

$$\boxed{r \simeq R - y\sin\theta}$$

This requires that the correction to the phase kr appearing in $\exp(ikr)$ due to the $y^2/2R$ term is small or

$$\left(\frac{2\pi}{\lambda}\right)\frac{1}{2}\frac{y^2}{R} < \pi$$

Consider $y = \pm b/2$, then this becomes

$b^2/4\lambda R \leq 1$, or

Fresnel #

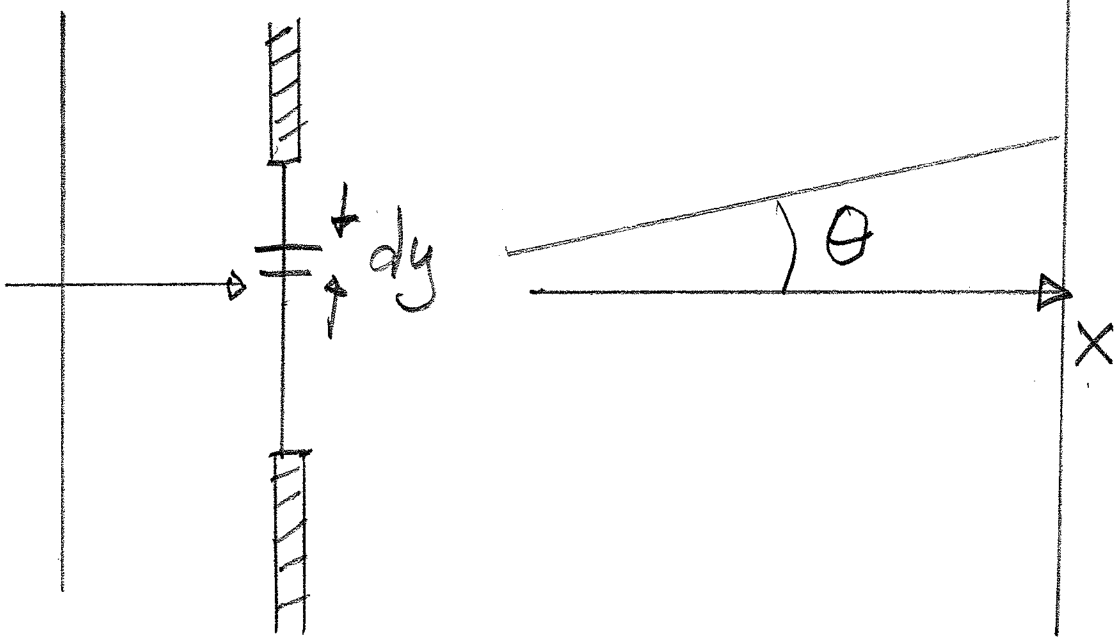
$$R > \frac{(b/2)^2}{\lambda}, \quad N_F = \frac{(b/2)^2}{\lambda R} < 1.$$

Thus with $R \sim x$ the distance to the observation screen

$R < \dfrac{(b/2)^2}{\lambda}$ — Fresnel region

$R > \dfrac{(b/2)^2}{\lambda}$ — Fraunhofer diffraction.

The Single slit (Hecht 10.2).



Consider the summation of the spherical waves with centers within a distance $dy <<< \lambda$ around y: These spherical waves will almost be identical so the resultant field dE is

$$dE \sim \frac{\mathcal{E}_L dy}{r} e^{ikr} \quad (\equiv dE(\vec{r},\omega))$$

(Here we use the complex representation whereas Hecht uses the real repres.)

$\mathcal{E}_L$ controls the strength of the source over the aperture.

In the Fraunhofer region

$$r \simeq R - y\sin\theta \sim R$$

Correction term more important in the phase so

$$dE \sim dy\, \mathcal{E}_L \left(\frac{e^{ikR}}{R}\right) e^{-iky\sin\theta}$$

and the field due to the whole slit

$$E(\vec{r},\omega) = \mathcal{E}_L \frac{e^{ikR}}{R} \int_{-b/2}^{b/2} dy\, e^{-iky\sin\theta}$$

$$= \mathcal{E}_L \frac{e^{ikR}}{R} \cdot \frac{(e^{-i\beta} - e^{i\beta})}{-ik\sin\theta}, \quad \beta = \left(\frac{kb}{2}\right)\sin\theta.$$

$$= \mathcal{E}_L \left(\frac{b e^{ikR}}{R}\right) \frac{\sin(\beta)}{\beta}$$

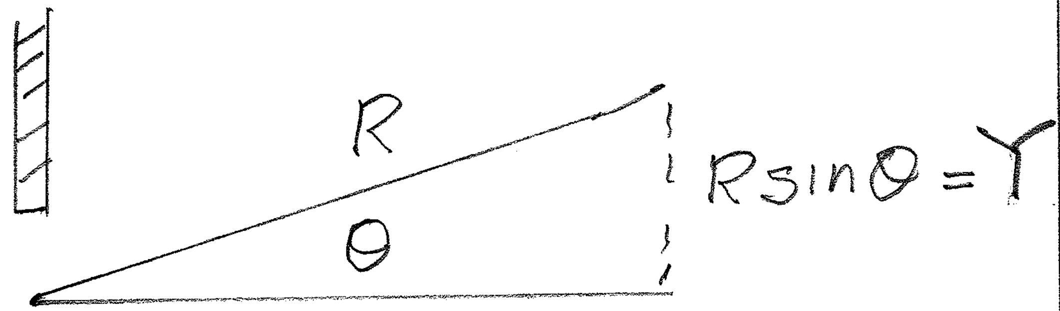
modulation due to aperture.

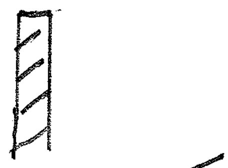
main spherical wave

input field strength

$$\frac{\sin(\beta)}{\beta} = \operatorname{sinc}(\beta) = \operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)$$

sinc- function.





$(I \propto |E|^2)$

$$E(\theta,\omega) = E(0,\omega)\operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)$$

$$\frac{I(\theta,\omega)}{I(0,\omega)} = \operatorname{sinc}^2\left(\frac{kb}{2}\sin\theta\right)$$

see next page for plots.

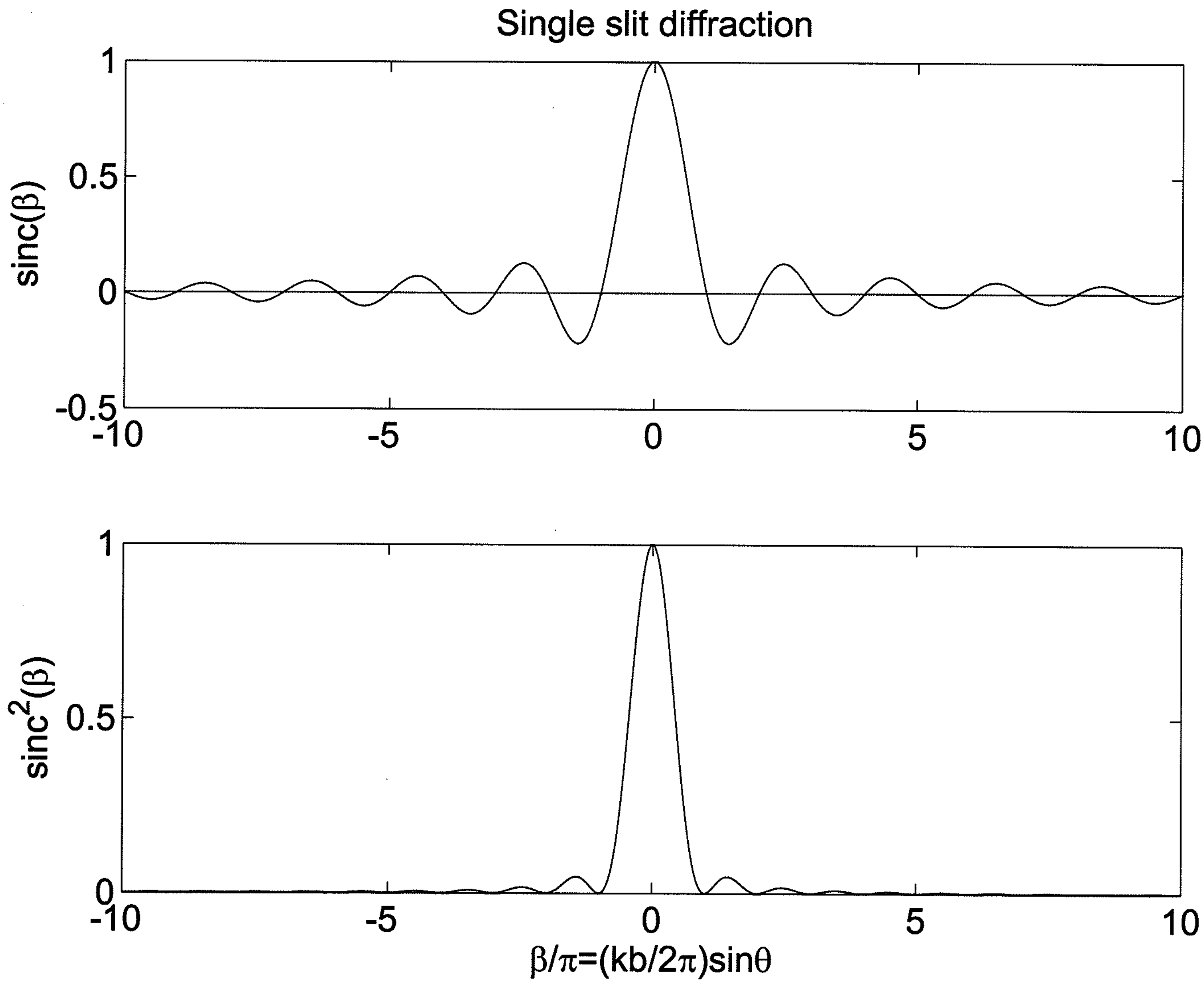
In the Fraunhofer region $\sin\theta \sim \theta$. Zeros of field occur at

$$\frac{kb}{2}\theta_m \sim m\pi, \quad m = \pm 1, \pm 2$$

Peak on-axis $\theta = 0$

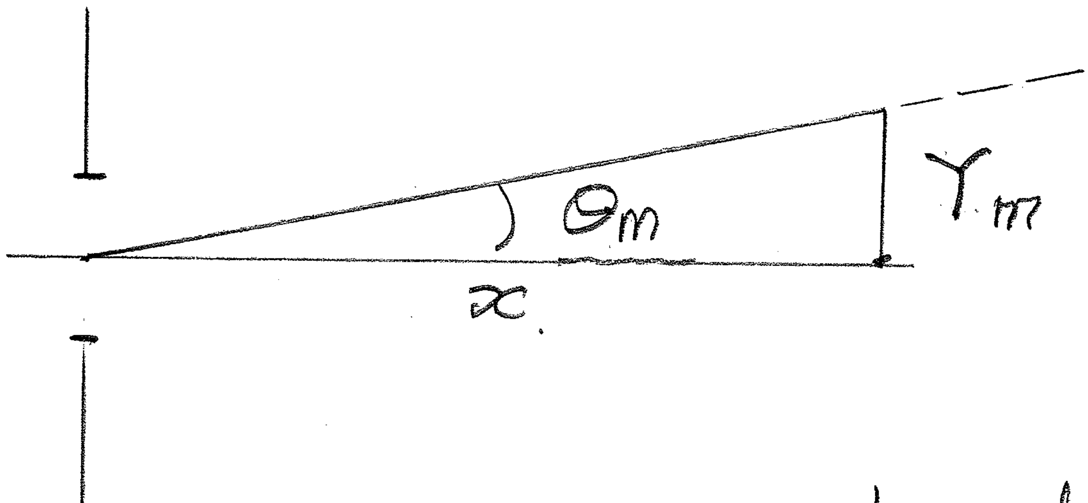
Nulls at $\beta_m = m\pi$, or

$$\frac{kb}{2}\sin\theta_m \simeq \frac{kb}{2}\theta_m \simeq m\pi$$



or

$$\theta_m \sim m\left(\frac{\lambda}{b}\right), \quad m = \pm 1, \pm 2 \ldots$$



Now the displacement y of the null from the x-axis obeys

$$\tan\theta_m = \frac{Y_m}{X} << 1$$

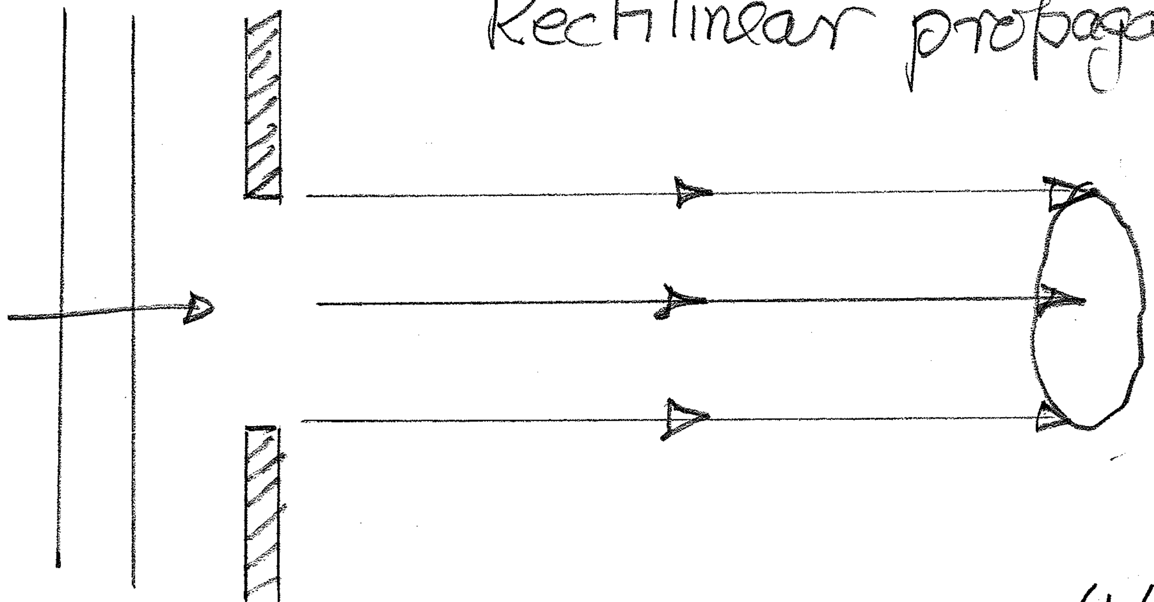
or

$$Y_m \sim X\theta_m \sim mX\left(\frac{\lambda}{b}\right).$$

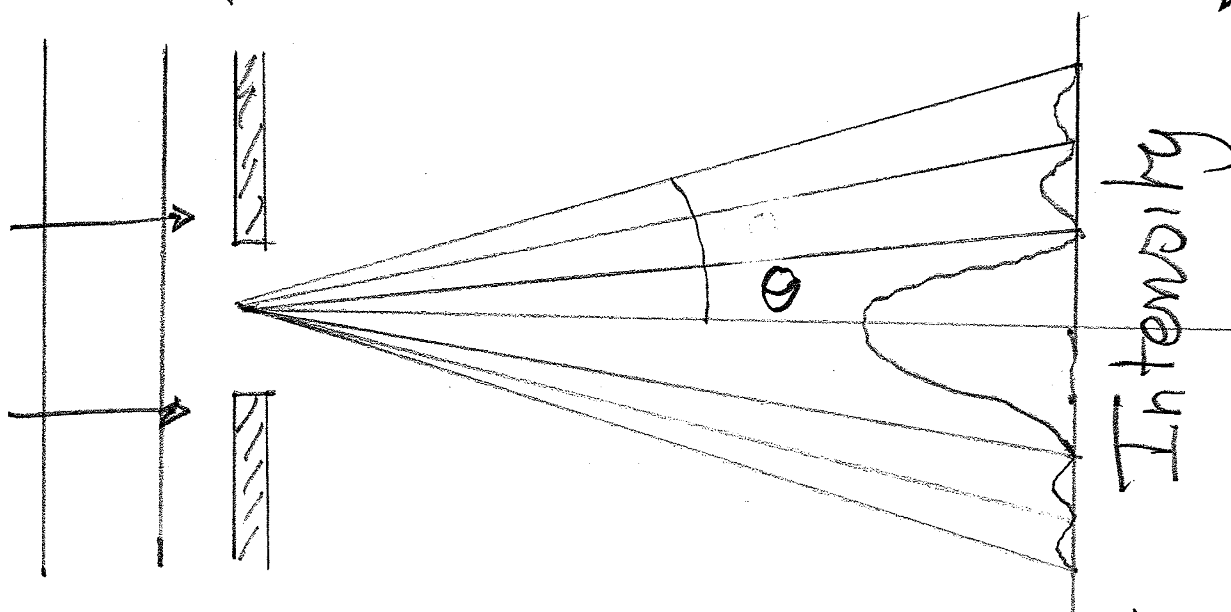
In the Fraunhofer region the diffraction pattern of the single slit is a sinc-function which expands in spatial extent as the observation distance increases - See applets

Geometric optics:

Rectilinear propagation

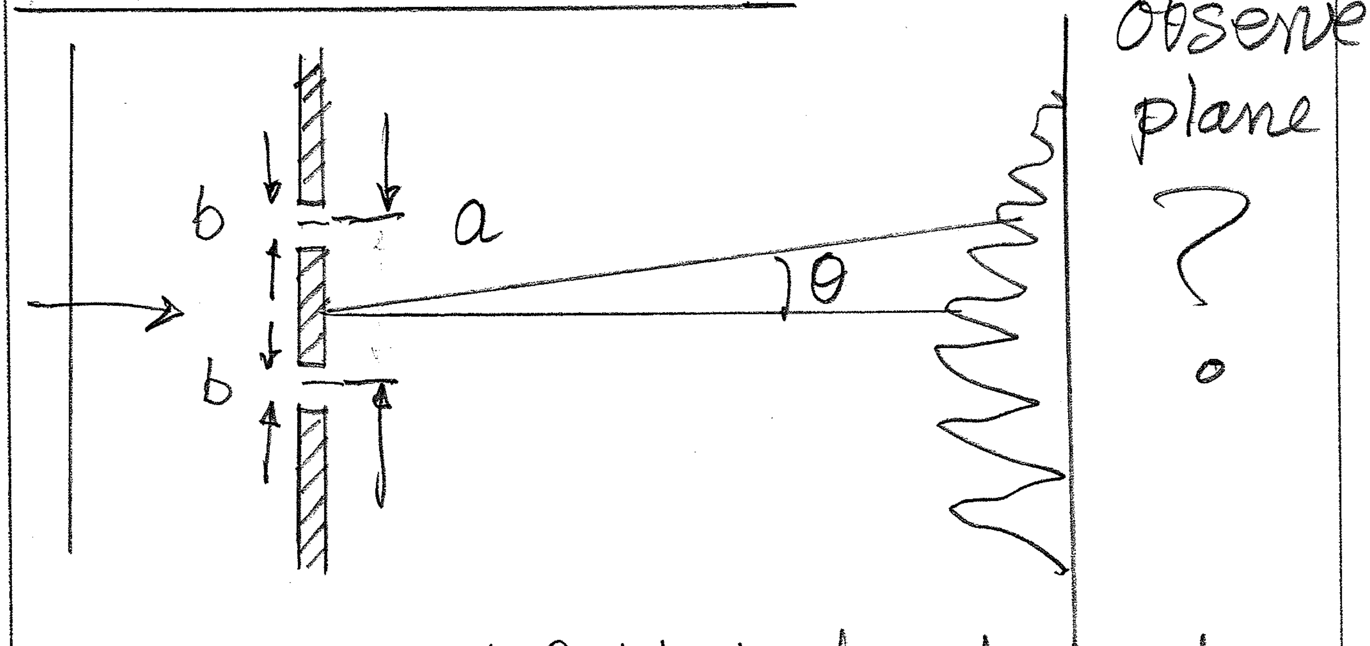


Wave optics (Fraunhofer $x > \frac{(b/2)^2}{\lambda}$)



Expanding center lobe + oscillating structure. See applets

Double slit diffraction



Now the total field at the observation plane is in the Fraunhofer region.

$$E(\vec{r},\omega) = \mathcal{E}_L \frac{e^{ikR}}{R} \int_{a/2-b/2}^{a/2+b/2} dy\, e^{-iky\sin\theta}$$

$$+ \mathcal{E}_L \frac{e^{ikR}}{R} \int_{-a/2-b/2}^{-a/2+b/2} dy\, e^{-iky\sin\theta}.$$

So there are contributions from each slit!

Performing the integrals yields.

$$E(\vec{r},\omega) = 2\mathcal{E}_L\left(\frac{b e^{ikR}}{R}\right) \operatorname{sinc}(\beta)\cos(\alpha)$$

where

$$\alpha = \frac{ka}{2}\sin\theta \ , \ \beta = \frac{kb}{2}\sin\theta.$$

or

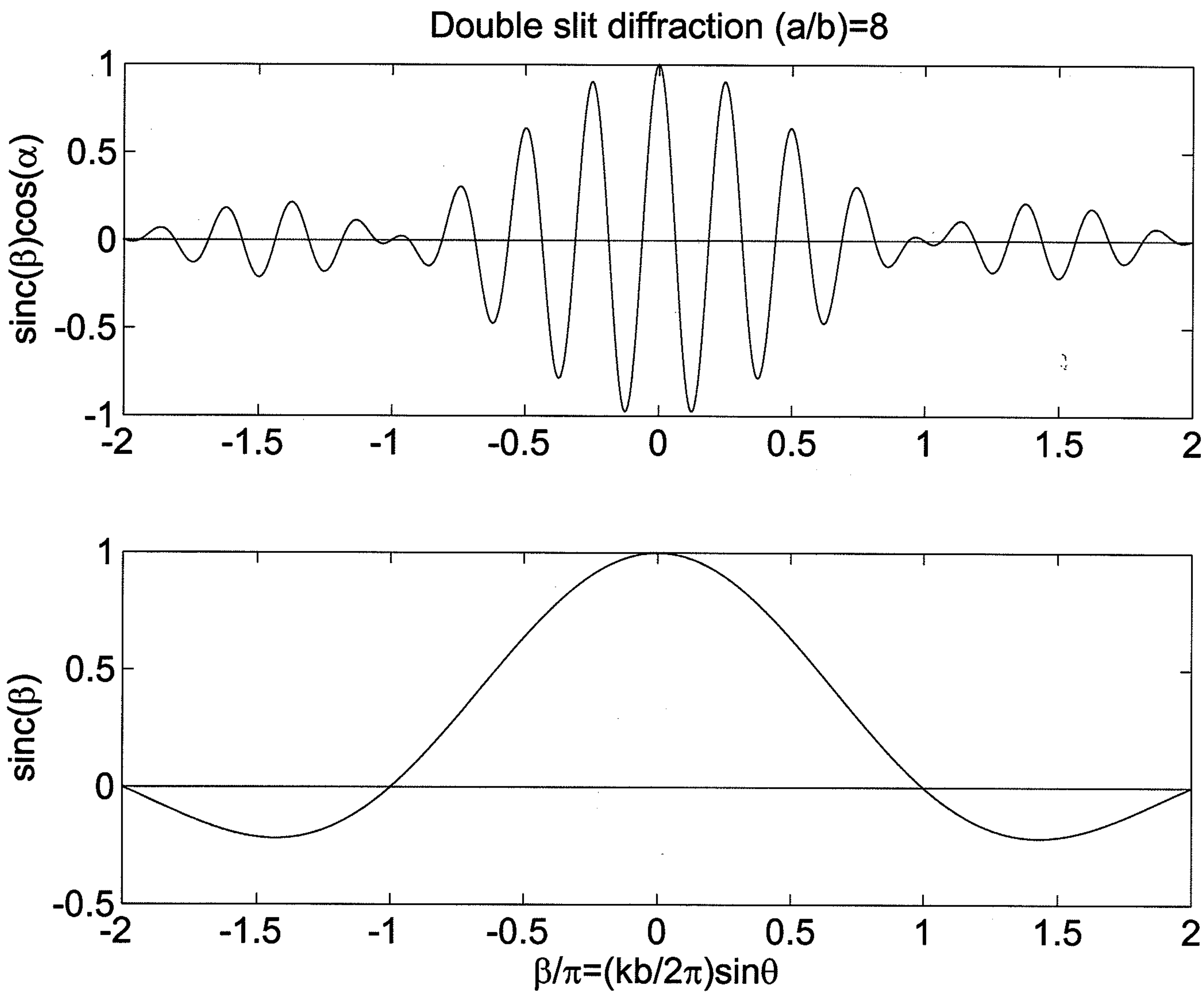
$$E(\theta,\omega) = 2E(0,\omega)\operatorname{sinc}(\beta)\cos(\alpha)$$

$$I(\theta,\omega) = 4I(\theta=0)\operatorname{sinc}^2(\beta)\cos^2(\alpha).$$

$I(\theta=0)$ peak intensity for a single beam. See plots for an example with $(a/b)=8$, & applet

This corresponds to Young's double slit expt. Easily extended to multiple slits (Hecht 10.2.3)

$$\alpha = \left(\frac{a}{b}\right)\beta .$$

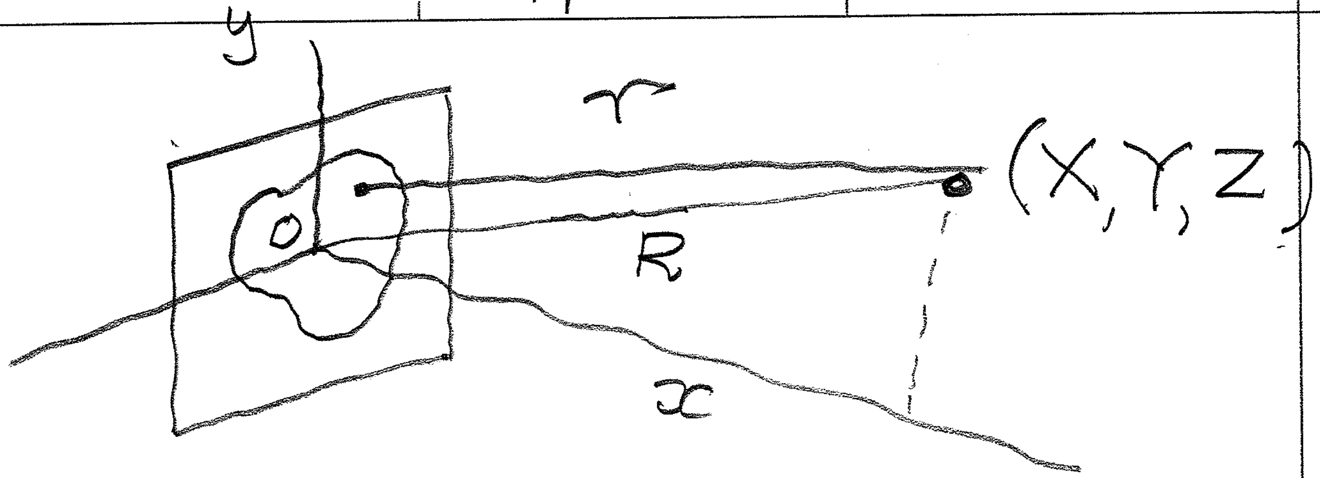


Two-dimensional diffraction.

So far we have tacitly looked at diffraction in one-dimension (y) which is transverse to the direction of propagation x: this is 1D diffraction. In general we have diffraction in two transverse dimensions y & z, or 2D diffraction, which we now examine.

Huygens-Frensnel principle

The Huygens-Fresnel principle is as before, namely each point on a wavefront acts as a secondary source of spherical waves. Then for a planar aperture or object we need to sum the spherical waves over the 2D aperture opening ($E=0$ outside)



(y, z) are transverse coordinates on the plane of the aperture

(X, Y, z) are transverse coordinates on the observation plane.

Then each spherical wave has the form

$$\vec{E}(\vec{r}, \omega) = \hat{e} \, \frac{\mathcal{R}}{r} e^{ikr}$$

$$= \hat{e} \, E(\vec{r}, \omega)$$

Hereafter we look at the scalar complex field $E(\vec{r}, \omega)$.

The distance r is given by $(x \equiv X)$

$$r = \sqrt{X^2 + (Y-y)^2 + (Z-z)^2},$$

$$R = \sqrt{X^2 + Y^2 + Z^2}.$$

Assuming

$$\sqrt{y^2+z^2}, \sqrt{Y^2+Z^2} \ll R$$

$$r = R\sqrt{1 + \frac{(y^2+z^2)}{R^2} - \frac{2(yY+zZ)}{R^2}}$$

$$\simeq R - \frac{(yY+zZ)}{R} + \frac{(y^2+z^2)}{2R}.$$

In the Fraunhofer region we assume the phase-shift due to the last term is small

$$\left(\frac{2\pi}{\lambda}\right) \cdot \frac{(y^2+z^2)}{2R} < \pi$$

giving the approximation

$$r \simeq R - \frac{(yY + zZ)}{R}$$

Thus for a spherical wave centered at (y,z) on the aperture plane

$$E(\vec{r},\omega) \simeq \mathcal{A}\,\frac{e^{ikR}}{R}\,e^{-ik(yY+zZ)/R}.$$

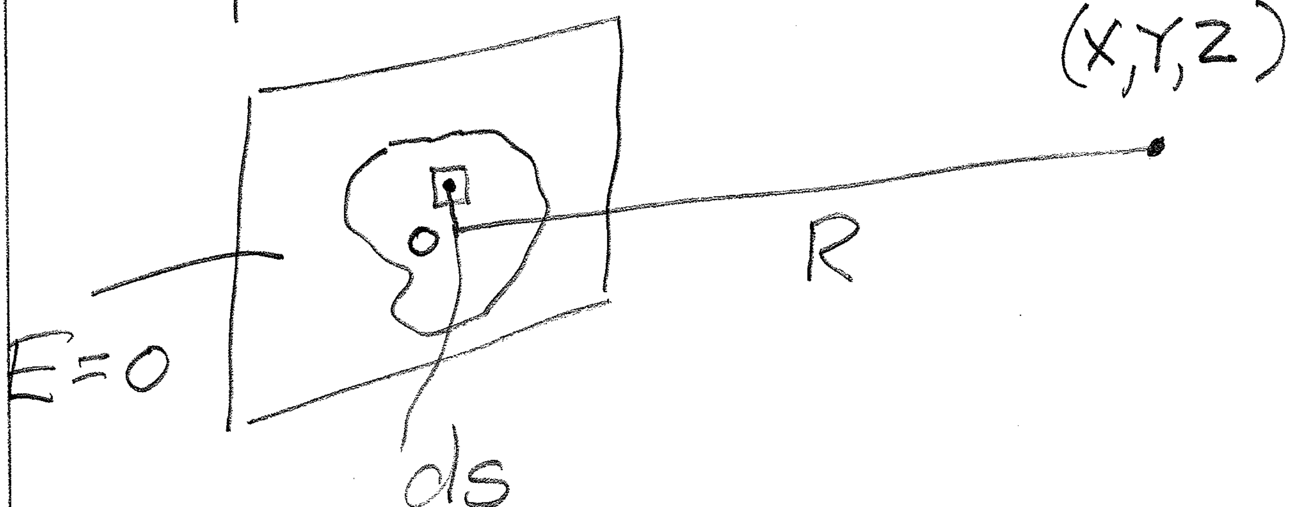
Consider a small patch of area $dS = dydz$ on the aperture plane over which the spherical waves vary little. Then the net field due to the patch can be written

$$dE = \frac{\mathcal{E}_A e^{ikR}}{R}\,e^{-ik(yY+zZ)/R}\,dS$$

The total field @ (X,Y,Z) due to the whole aperture is then

$$E(X,Y,Z) = \frac{\mathcal{E}_A e^{ikR}}{R} \int_{\text{Aperture}} ds\, e^{-ik(yY+zZ)/R}$$

where the integral is over the aperture opening, the field being zero outside the aperture



The above formula is the main result of this section and expresses the field in the Fraunhofer region as an integral over the aperture. We shall apply this to rectangular & circular apertures below.

Notes

- The Fraunhofer region is often called the far field

$$\left(\frac{2\pi}{\lambda}\right)\cdot\frac{(y^2+z^2)}{2R} < \pi$$

- The Fresnel region is often called the near field

$$\left(\frac{2\pi}{\lambda}\right)\frac{(y^2+z^2)}{2R} > \pi$$

- Here we assume a uniformly illuminated aperture.
- The far field profile is related to the Fourier transform of the aperture (Fourier optics)

$$E(X,Y,Z) \propto \int dy \int dz\, e^{-ik(yY+zZ)/R}$$

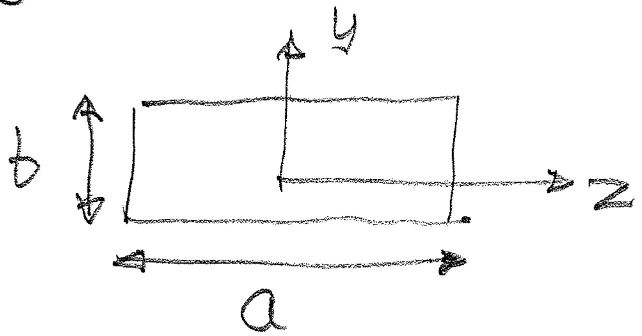
$$\propto \int dy \int dz\, e^{-i(k_y y + k_z z)} \qquad \boxed{\text{2D F.T.}}$$

$$k_y = +kY/R, \quad k_z = +kZ/R.$$

Rectangular aperture.

In this case the aperture opening is described by the integration domain.

$$-b/2 \leq y \leq b/2 \,, \quad -a/2 \leq z \leq a/2$$



$$A = ab$$

Then we get the far field profile.

$$E(\vec{r},\omega) = \frac{\mathcal{E}_A e^{ikR}}{R} \int_{-b/2}^{b/2} dy \int_{-a/2}^{a/2} dz \, e^{-ik(yY+zZ)/R}$$

$$= \frac{\mathcal{E}_A e^{ikR}}{R} \int_{-b/2}^{b/2} dy \, e^{-ikyY/R} \int_{-a/2}^{a/2} dz \, e^{-ikzZ/R}$$

Performing the integrals yields

$$E(X,Y,Z) = \frac{A\mathcal{E}_A e^{ikR}}{R} \operatorname{sinc}(\alpha')\operatorname{sinc}(\beta')$$

where $A = ab$ is the area of the aperture, and

$$\alpha' = \frac{kaZ}{2R}, \quad \beta' = \frac{kbY}{2R}.$$

This gives the intensity $(I \propto |E|^2)$.

$$I(Y,Z) = I(0)\operatorname{sinc}^2(\alpha')\operatorname{sinc}^2(\beta').$$

Thus a 2D rectangular aperture gives rise to a far field profile that is like the product of 2 1D slits, one along Y, one along Z.

The nulls in the field occur at $\alpha'_m = m\pi$, $\beta'_m = m\pi$, or

$$Z_m = m\left(\frac{\lambda R}{a}\right), \quad Y_m = m\left(\frac{\lambda R}{b}\right)$$

Thus if $a > b$, the interference fringes along Z are of shorter period than those along Y — for a square aperture they are the same.

See Hecht Fig. 10.20 p 466, and Wyant's Mathematica webpage.

We finish by again stressing that according to geometric optics the diffracted field would simply be the incident plane-wave truncated by the aperture. Reality is much more interesting!!

Circular aperture

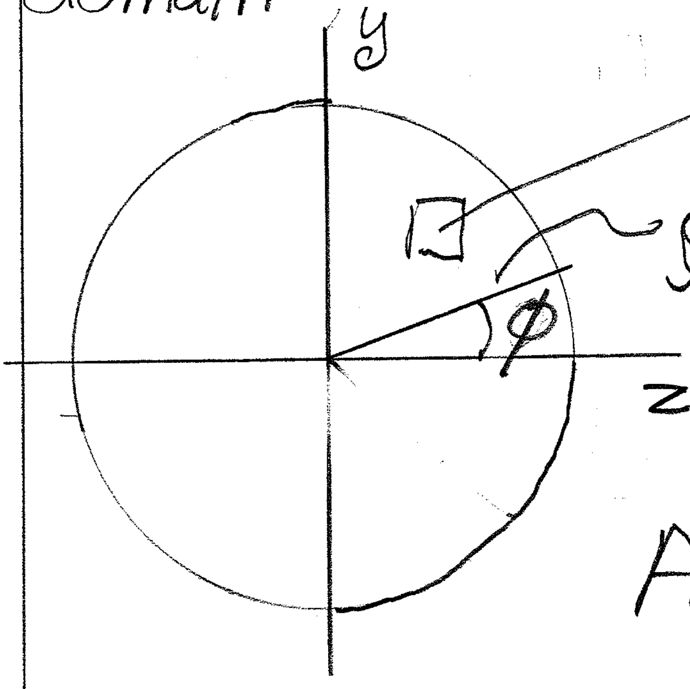
In this case the aperture opening is described by the integration domain

$$y^2 + z^2 \leq a^2$$

for a circular aperture of radius 'a.'
Again the field field profile is given by

$$E(\vec{r},\omega) = \frac{\mathcal{E}_A e^{ikR}}{R} \iint_{\text{Aperture}} dS\, e^{-ik(yY+zZ)/R}.$$

but with a different integration domain



$$dS = dy\,dz$$

$$= \rho\, d\rho\, d\phi$$

$$A = \pi a^2$$

It proves useful to transform to cylindrical coordinates on the aperture & observation planes.

$$z = \rho\cos\phi\,, \quad y = \rho\sin\phi$$

$$Z = q\cos\Phi\,, \quad Y = q\sin\Phi$$

$$dS = dy\,dz = \rho\,d\rho\,d\phi$$

This yields the far field profile.

$$E(\vec{r},\omega) = \frac{\mathcal{E}_A e^{ikR}}{R}\int_0^a d\rho\,\rho\int_0^{2\pi} d\phi\, e^{-\frac{ik\rho q}{R}\cos(\phi-\Phi)}.$$

This integral can be evaluated in terms of special functions called Bessel functions that in all likelyhood you have not encountered yet. However, they can be evaluated in Matlab and we shall explore them this way.

Here I state the profile in the far field without proof, the details are given in Hecht 10.2.5 pp. 467-469.

$$E(q) = \left(\frac{A\,\mathcal{E}_A e^{ikR}}{R}\right) \cdot \frac{2J_1(kaq/R)}{(kaq/R)}$$

main spherical wave

modulation due to aperture

where $J_m(x)$ is a Bessel function of the first kind of order m.

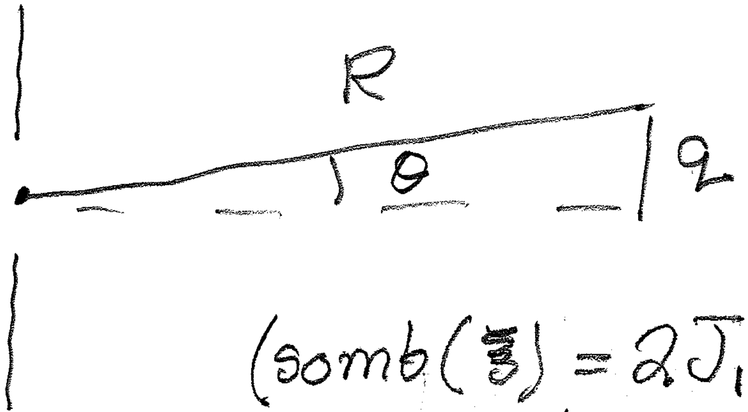
Notes

- q is the radial coordinate in the far field.
- $2J_1(x)/x$ 2D analogue of $\sin(x)/x$.
- Intensity profile ($I \propto |E|^2$)

$$I(q) \sim I(0) \cdot \left[2J_1(kaq/R)/(kaq/R)\right]^2$$

Sombrero function

- Using $\sin\theta = q/R$

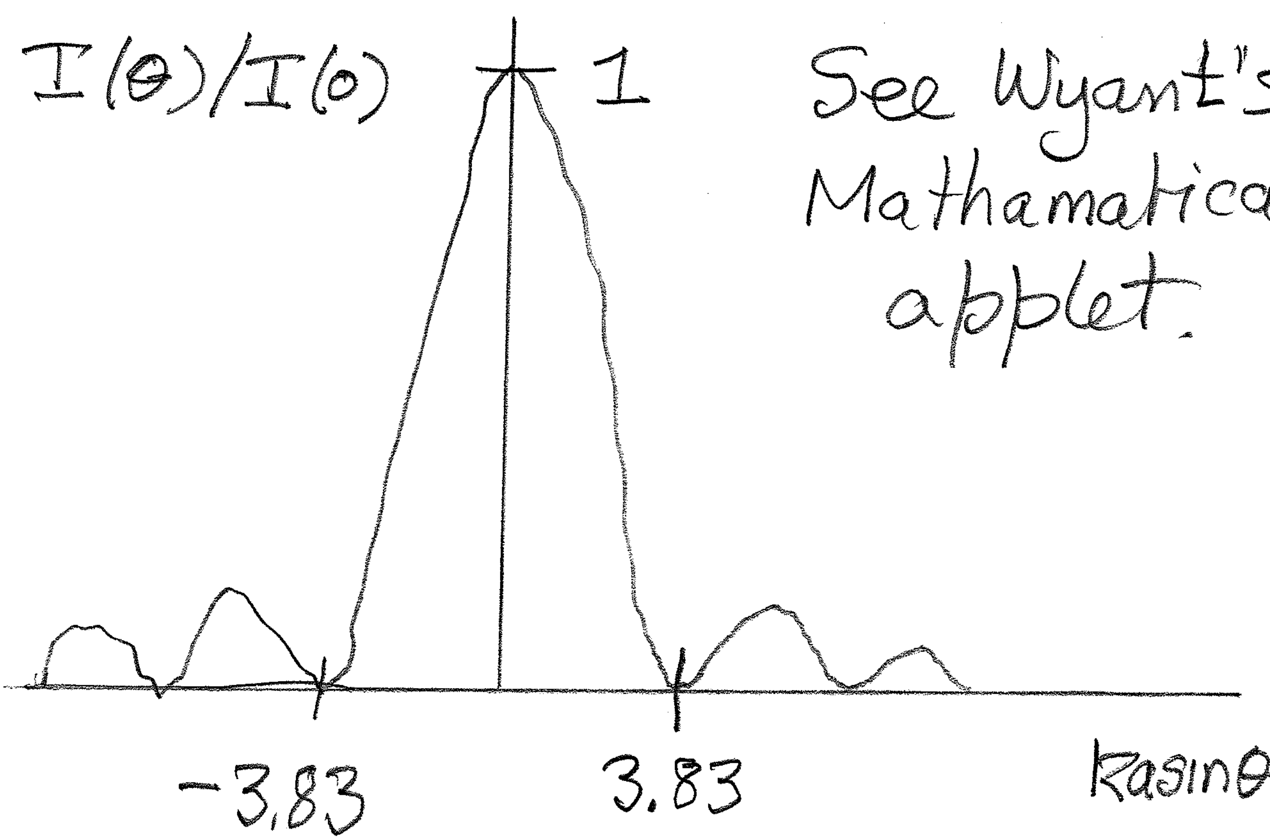


$(\mathrm{somb}(\xi) = 2J_1(\pi\xi)/(\pi\xi))$

we can write.

$$I(\theta) = I(0)\left[\frac{2J_1(Ra\sin\theta)}{Ra\sin\theta}\right]^2$$

- General form (use Matlab plot)



See Wyant's Mathamatica applet.

First zero when $J_1(u) = 0$, $u = 3{,}83$, or

$$\frac{k a q_1}{R} = 3{,}83$$

giving

$$\boxed{q_1 = 1{,}22 \frac{R\lambda}{2a}}$$

This is the radius of the Airy disk, a bright central maximum surrounded by rings. Note, the Airy disk expands with R.

- The Airy disk sets the resolution limit of optical imaging systems with limiting apertures. The size of the Airy disk gives limiting resolution with $R = f$ for a lens of finite size a. See Hecht 10.2.6.