# 3D wave in a waveguide: Phase velocity

(Optional material beyond syllabus of OPTI-310)

**Example:** This is to illustrate:
a) construction of spatially localized solutions to 3D wave equation
b) phase velocity in a waveguide can be larger than $c$

**Physical context:** Propagation of waves inside a waveguide with square cross-section. Let the waveguide be oriented along $z$ and its side be of length $a$. We assume that the material of the waveguide is a perfect conductor. As such it will "force" the EM field to be zero at the wall of the waveguide. To keep everything simple, we will consider the wave to be *scalar* (i.e. not a vector like in a real EM field).

**Mathematical formulation:**
We need to find a 3D scalar solution to the wave equation inside the waveguide, i.e. for

$$z \in (-\infty, +\infty) \quad |x| < a/2 \quad |y| < a/2$$

and such that this solution vanishes at waveguide walls:

$$[\Delta - \frac{1}{c^2}\partial_{tt}]\psi(x, y, z, t) = 0 \qquad \psi(x = \pm a/2, y, z, t) = 0 \qquad \psi(x, y = \pm a/2, z, t) = 0$$

Assume that waves propagate with velocity $c$ in the medium inside the waveguide, and that wavelength $\lambda$ and/or $\omega$ is given.

**A)** Show that

$$\psi = \cos[k_\perp x] \cos[k_\perp y] \exp[i(k_z z - \omega t)]$$

can represent this solution. What condition must $k_\perp$ and $k_z$ obey?

**Answer:**

Insert the ansatz (parameterized guess) into the wave equation to get:

$$[\Delta - \frac{1}{c^2}\partial_{tt}]\psi = (-2k_\perp^2 - k_z^2 + \frac{\omega^2}{c^2})\psi = 0$$

Since $\psi$ is not identical zero, it must be that

$$k_z^2 + 2k_\perp^2 = \omega^2/c^2 = (2\pi/\lambda)^2$$

This is the dispersion relation imposed by WE.

Next, we need to satisfy the boundary conditions:

$$\psi(x, y = \pm a/2, z, t) = \psi(x = \pm a/2, y, z, t) = 0$$

One solution to this is to require that:

$$\cos[k_\perp(\pm a)/2] = 0 \qquad \frac{a}{2}k_\perp = \frac{\pi}{2} \qquad k_\perp = \frac{\pi}{a}$$

Insert this into the dispersion relation to solve for $k_z$:

$$k_z = \sqrt{\frac{\omega^2}{c^2} - 2k_\perp^2} = \sqrt{\frac{\omega^2}{c^2} - \frac{2\pi^2}{a^2}}$$

**B)** What is the phase velocity $v_p$ of this wave along the $z$ direction? Is it larger or smaller than the wave velocity $c$? Express your result in terms of $c, a, \lambda$. Discuss how $v_p$ depends on:

i) $\omega$ (or wavelength)

ii) transverse dimensions of the waveguide

**Answer:**

The phase velocity can be obtained from the last term in the solution:

$$\exp[i(k_z z - \omega t)] = \exp[ik_z(z - \frac{\omega}{k_z}t)] = \exp[ik_z(z - v_p t)]$$

Which means that the velocity of phase fronts along $z$ is:

$$v_p = \frac{\omega}{k_z} = \frac{c}{\sqrt{1 - \frac{2\pi^2 c^2}{\omega^2 a^2}}} = \frac{c}{\sqrt{1 - \frac{\lambda^2}{2a^2}}}$$

So we see that the phase velocity is larger than $c$. This is in no contradiction with relativity: Phase fronts are purely geometric objects which are allowed to move faster than $c$ because they do not transfer energy or information. $v_p$ is *not* the "signaling velocity."

We can also see that the deviation from $c$ is larger in a smaller waveguide.

**Note:** This is the simplest possible example of dispersion (i.e. dependence of the phase velocity on the frequency) caused by a waveguide. We speak of "waveguide dispersion"...

# 3D wave in a waveguide: Group velocity

(Optional material beyond syllabus of OPTI-310)

**Example continued:** This is continuation of a previously formulated example where we explored the 3D wave equation solution

$$\psi(\omega) = \cos[k_\perp x] \cos[k_\perp y] \exp[i(k_z z - \omega t)]$$

for

$$z \in (-\infty, +\infty) \quad |x| < a/2 \quad |y| < a/2$$

and PEC boundary condition (i.e. vanishing solution at $|x| = a/2$ and $|y| = a/2$).
In what follows we assume $a >> \lambda$ and will use this for certain approximations.

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**Your task:** is to show that a wavepacket formed from a superposition of the above solutions with angular frequencies in the **narrow** band $\omega_0 \pm \Delta\omega$ can be found to have the form

$$A = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \psi(\omega) d\omega = M(x,y) \frac{\sin[\Delta\omega(z/v_g - t)]}{(z/v_g - t)} \exp[i\omega_0(z/v_p - t)]$$

This is written such that the expression shows that $v_p$ and $v_g$ are velocities of the carrier wave and of the envelope, respectively.

You will also show that the phase and group velocities can be approximated by

$$\frac{1}{v_p} = \frac{1}{c} - \frac{ck_\perp^2}{\omega_0^2} \qquad \frac{1}{v_g} = \frac{1}{c} + \frac{ck_\perp^2}{\omega_0^2}$$

## Solution

$$A = \int_{\omega_0-\Delta\omega}^{\omega_0-\Delta\omega} \psi(\omega)d\omega = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \cos[k_\perp x]\cos[k_\perp y]\exp[i(k_z(\omega)z - \omega t)]d\omega$$

where $k_z$ must keep the dispersion relation preserved so that

$$k_z(\omega) = \sqrt{\omega^2/c^2 - 2k_\perp^2}$$

and we have established in the first part of this problem that

$$\cos[k_\perp a/2] = 0 \quad k_\perp = \pi/a$$

(this gives us the fundamental mode of the waveguide).

To calculate the integral, substitute:

$$\omega = \omega_0 + w$$

$$A = \cos[k_\perp x]\cos[k_\perp y]\int_{-\Delta\omega}^{+\Delta\omega} \exp[i(k_z(\omega_0+w)z - (\omega_0+w)t)]dw$$

**First approximation:** Take advantage of the fact that $a >> \lambda$ and therefore $k_\perp << \omega/c$. We can therefore approximate

$$k_z(\omega) = \sqrt{\omega^2/c^2 - 2k_\perp^2} \approx \frac{\omega}{c} - \frac{ck_\perp^2}{\omega}$$

**Second approximation:** Take advantage of the fact that $w$ is small in comparison to $\omega_0$:

$$k_z(\omega) \approx \frac{\omega}{c} - \frac{ck_\perp^2}{\omega} \approx \frac{\omega_0 + w}{c} - \frac{ck_\perp^2}{\omega_0}(1 - \frac{w}{\omega_0})$$

This is linear in $w$ so we should be able to integrate without difficulty... Insert the above $k_z(\omega)$ into integral...

$$A = \cos[k_\perp x]\cos[k_\perp y]\int_{-\Delta\omega}^{+\Delta\omega} \exp[i((\frac{\omega_0 + w}{c} - \frac{ck_\perp^2}{\omega_0}(1 - \frac{w}{\omega_0}))z - (\omega_0 + w)t)]dw$$

... and collect $w$ in the exponential to get:

$$A = \cos[k_\perp x]\cos[k_\perp y]\exp i[\frac{\omega_0 z}{c} - \frac{ck_\perp^2}{\omega_0}z - \omega_0 t]\int_{-\Delta\omega}^{+\Delta\omega} \exp[i((\frac{w}{c} + \frac{ck_\perp^2}{\omega_0}\frac{w}{\omega_0}))z - wt)]dw$$

To simplify the expressions, introduce

$$\frac{1}{v_p} = \frac{1}{c} - \frac{ck_\perp^2}{\omega_0^2} \qquad\qquad \frac{1}{v_g} = \frac{1}{c} + \frac{ck_\perp^2}{\omega_0^2}$$

and see that

$$A = \cos[k_\perp x]\cos[k_\perp y]\exp[i\omega_0(\frac{z}{v_p} - t)]\int_{-\Delta\omega}^{+\Delta\omega} \exp[iw(\frac{z}{v_g} - t)]dw$$

Calculate the integral to obtain the final form

$$A = \cos[k_\perp x]\cos[k_\perp y]\exp[i\omega_0(\frac{z}{v_p} - t)]\frac{2\sin[\Delta\omega(\frac{z}{v_g} - t)]}{(\frac{z}{v_g} - t)}$$

So the final result is a product of

a) modal field $M(x,y) = 2\cos[k_\perp x]\cos[k_\perp y]$ — this is the transverse shape of the wave

b) envelope $\sin[\Delta\omega(z/v_g - t)]/(\frac{z}{v_g} - t)$ — this moves with the group velocity $v_g$

c) and carrier wave $\exp[i\omega_0(\frac{z}{v_p} - t)]$ — this moves with the phase velocity $v_p$

## Take away messages:

- The phase velocity is larger than the natural speed of wave (light).
- However, the group velocity is smaller.
- This means no conflict with speed of light being the ultimate limit to speed. This is because phase of the carrier does not carry useful information, and as such it is not subject to the limit.
- On the other hand, the wave packet can be used for signaling, but it propagates slower than $c$.
- Note than (in the given approximation) the average of velocities $(v_p + v_g)/2 = c$
- You can check that $v_p$ from the first part can be approximated by $v_p$ found in the second part.
- The envelope depends on the *spread* of parameters: $\Delta\omega$
- The carrier depends on the *average* of parameters: $\omega_0$