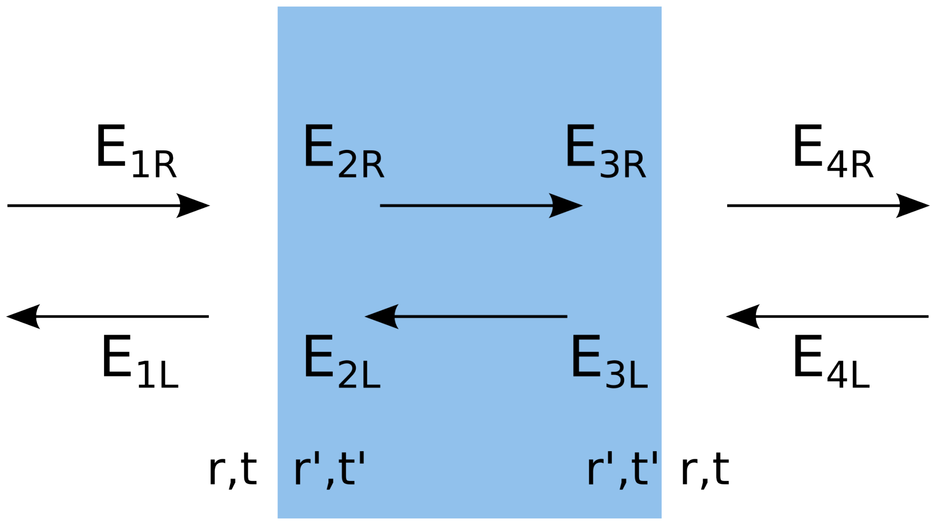
Alternative derivations for layered structures: Transfer matrix calculus

We have derived the following expression in the class

$$E_T = E_0 \frac{t^2}{1 - r^2 e^{i\delta}}$$

for the transmitted amplitude in the Fabry-Perot geometry. Here we want to avoid the consideration based on partial waves.

Instead, represent the solution as a superposition of plane waves propagating in both directions (index R, and L for right and left, respectively), and look for their amplitudes at locations indicated in the picture:



In complete analogy to the case of a single material interface, we must satisfy the boundary conditions — that is how we determine the unknown amplitudes of the forward and backward waves.

Start with the first material interface and TE polarization when the electric field must be continuous:

$$E_{2R} = tE_{1R} + r'E_{2L} \qquad E_{1L} = t'E_{2L} + rE_{1R}$$

Pretend we know field amplitudes with index 1, and solve for those with index 2. Use the relations we derived previously,

$$tt' = 1 - r^2 \qquad r' = -r \quad ,$$

and write the field amplitudes E_2 (in the middle region) in this matrix form:

$$\begin{pmatrix} E_{2R} \\ E_{2L} \end{pmatrix} = M_{TE}^{(12)} \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix} \equiv \begin{pmatrix} 1/t' & -r/t' \\ -r/t' & 1/t' \end{pmatrix} \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix}$$

M_{TE} is the transfer matrix (for TE polarization) that relates the field amplitudes in medium 1 to medium 2, or "transfers" them from one side of interface to the other.

Next we need a similar matrix for a length of homogeneous material (to represent the inside of FP).

$$\begin{pmatrix} E_{3R} \\ E_{3L} \end{pmatrix} = M_{SLAB} \begin{pmatrix} E_{2R} \\ E_{2L} \end{pmatrix} \equiv \begin{pmatrix} e^{ik_z d} & 0 \\ 0 & e^{-ik_z d} \end{pmatrix} \begin{pmatrix} E_{2R} \\ E_{2L} \end{pmatrix}$$

Important: k_z is the z-component of the wave-vector in the medium between the interfaces:

$$k_z = \frac{2\pi}{\lambda} n \cos\theta_t \qquad k_z d \equiv \phi = \delta/2$$

The last is the matrix for the "output interface". We can obtain it from the first interface matrix, M_{TE} by changing $r \to r'$ and $t \to t'$ (this only holds for our situation when the first and last media are the same, of course):

$$\begin{pmatrix} E_{4R} \\ E_{4L} \end{pmatrix} = M_{TE}^{(21)} \begin{pmatrix} E_{3R} \\ E_{3L} \end{pmatrix} \equiv \begin{pmatrix} 1/t & -r'/t \\ -r'/t & 1/t \end{pmatrix} \begin{pmatrix} E_{3R} \\ E_{3L} \end{pmatrix}$$

The relation between the "input" and "output" amplitudes of the whole FP slab is then

$$\begin{pmatrix} E_{4R} \\ E_{4L} \end{pmatrix} = M_{FP} \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix} \equiv M_{TE}^{(21)} M_{SLAB} M_{TE}^{(12)} \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix}$$

Note: This is a similar technique as the Jones calculus, but these column vectors and matrices must not be confused with Jones!

Explicitly, the FP matrix is:

$$M_{FP} = \begin{pmatrix} \frac{e^{+i\phi}-e^{-i\phi}r^2}{1-r^2} & +\frac{(e^{+i\phi}-e^{-i\phi})r}{1-r^2} \\ -\frac{(e^{+i\phi}-e^{-i\phi})r}{1-r^2} & \frac{e^{-i\phi}-e^{+i\phi}r^2}{1-r^2} \end{pmatrix} \quad \text{and its inverse is:} \quad M_{FP}^{-1} = \begin{pmatrix} \frac{e^{-i\phi}-e^{+i\phi}r^2}{1-r^2} & -\frac{(e^{+i\phi}-e^{-i\phi})r}{1-r^2} \\ +\frac{(e^{+i\phi}-e^{-i\phi})r}{1-r^2} & \frac{e^{+i\phi}-e^{-i\phi}r^2}{1-r^2} \end{pmatrix}$$

Now we can extract the amplitude transmission coefficient. In our situation, E_{1R} is given as the incident amplitude. Further, we know that E_{4L} is zero (because no light is incident from the right). Thus,

$$\begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix} = M_{FP}^{-1} \begin{pmatrix} E_{4R} \\ E_{4L} \end{pmatrix} = M_{FP}^{-1} \begin{pmatrix} E_{4R} \\ 0 \end{pmatrix}$$

and

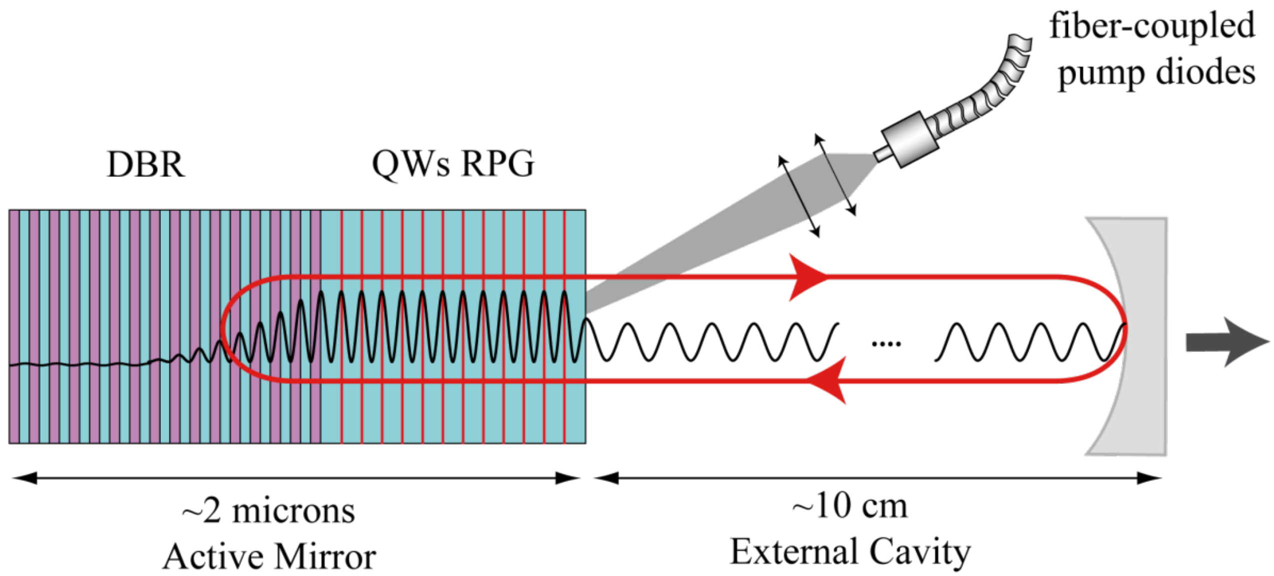
$$E_{1R} = \left(M_{FP}^{-1} \right)_{11} \; E_{4R}$$

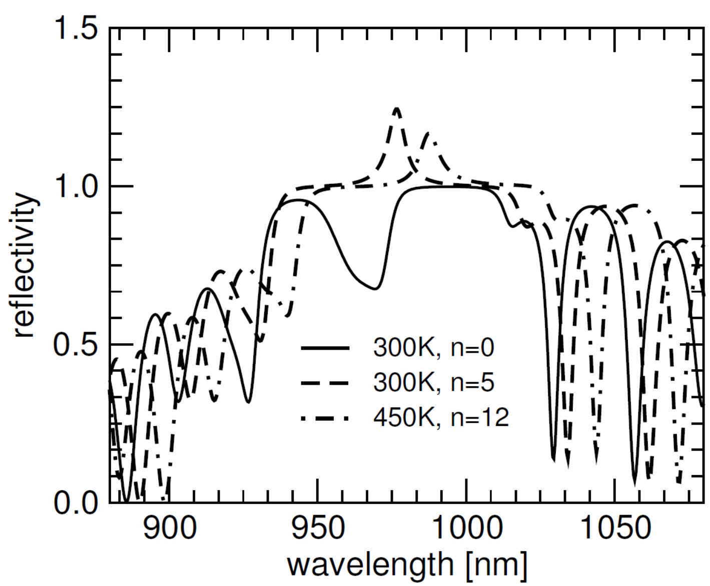
from where we simply read off the transmission coefficient as

$$t_{FP} = \frac{E_T}{E_0} = \frac{E_{4R}}{E_{1R}} = \frac{1}{\left(M_{FP}^{-1} \right)_{11}} = \frac{1 - r^2}{e^{-i\phi} - e^{+i\phi} r^2} = e^{+i\phi} \frac{1 - r^2}{1 - r^2 e^{2ik \cos\theta_t}} = e^{+i\phi} \frac{tt'}{1 - r^2 e^{i\delta}}$$

Apart from the phase pre-factor $e^{+i\phi}$ (which the method used in class simply dropped), this is the same result as the one obtained by summing up partial wave amplitudes.

Advantage: Transfer-matrix technique applies directly to arbitrary stratified structures.





Active mirror reflectivity: Depending on temperature and pumping, there can be net gain during the reflection. This kind of laser design requires accurate calculation of the reflection coefficient across a band of wavelengths and range of temperatures. Transfer matrix technique is most suitable.

Material interface revisited

Next, we figure out “elementary” matrices for bare material interfaces, both in the TE and TM polarizations. Then we will be able to apply the transfer matrix method to an arbitrary complicated structure.

Solution to the wave (or Maxwell) equations will be parametrized by vectorial amplitudes of forward (+) and backward (−) propagating plane-waves:

$$\vec{E}_1(z) = \left[\vec{E}_1^+ e^{+ik_z z} + \vec{E}_1^- e^{-ik_z z}\right] e^{i(k_x x + k_y y - \omega t)}$$

$$\vec{H}_1(z) = \left[\vec{H}_1^+ e^{+ik_z z} + \vec{H}_1^- e^{-ik_z z}\right] e^{i(k_x x + k_y y - \omega t)}$$

We will use index (here 1) to indicate the medium in which this expansion is applied.

Transfer matrix for a material interface

This time we want to express the transfer matrix of an "elementary" material interface, i.e. a sharp boundary between two media, but no intermediate layers no matter how thin...

TE case:

$$\begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = M_{TE} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

$$M_{TE} = \frac{1}{2} \begin{pmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{pmatrix} \qquad \alpha = \frac{k_z^{(1)}}{k_z^{(2)}}$$

TM case:

$$\begin{pmatrix} H_2^+ \\ H_2^- \end{pmatrix} = M_{TM} \begin{pmatrix} H_1^+ \\ H_1^- \end{pmatrix}$$

$$M_{TM} = \frac{1}{2} \begin{pmatrix} 1+\beta & 1-\beta \\ 1-\beta & 1+\beta \end{pmatrix} \qquad \beta = \frac{\epsilon_2 k_z^{(1)}}{\epsilon_1 k_z^{(2)}}$$

Relation to Fresnel formulas

The above relations are nothing but an equivalent statement of Fresnel equations for a simple material interface.

Exercise:

A) Starting from the above equations, derive Fresnel equations for a TE (s) polarization
B) Starting from the above equations, derive Fresnel equations for a TM (p) polarization

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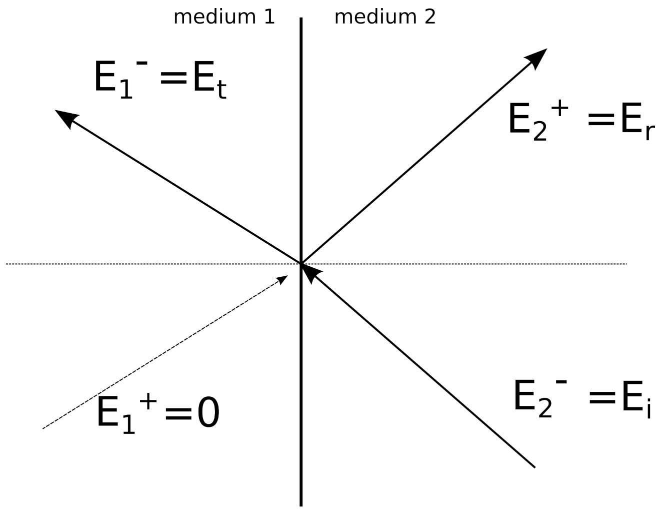
Solution A)

TE case:

This is possible to do in more than one way (of course). One efficient approach is to imagine that the incident wave comes from the medium 2:



$$\begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{pmatrix}\begin{pmatrix} E_1^+ = 0 \\ E_1^- \end{pmatrix} \qquad \alpha = \frac{k_z^{(1)}}{k_z^{(2)}}$$

$$\begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = \frac{E_1^-}{2}\begin{pmatrix} 1-\alpha \\ 1+\alpha \end{pmatrix}$$

$$r_s = \frac{E_r}{E_i} = \frac{E_2^+}{E_2^-} = \frac{1-\alpha}{1+\alpha} = \frac{k_2-k_1}{k_2+k_1} = \frac{n_2\cos\theta_2 - n_1\cos\theta_1}{n_2\cos\theta_2 + n_1\cos\theta_1} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_s = \frac{E_t}{E_i} = \frac{E_1^-}{E_2^-} = \frac{2E_1^-}{(1+\alpha)E_1^-} = \frac{2k_2}{k_2+k_1} = \frac{2n_2\cos\theta_2}{n_2\cos\theta_2 + n_1\cos\theta_1} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

Solution B)

TM case (using the same frame of reference):

$$\begin{pmatrix} H_2^+ \\ H_2^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+\beta & 1-\beta \\ 1-\beta & 1+\beta \end{pmatrix} \begin{pmatrix} H_1^+ = 0 \\ H_1^- \end{pmatrix} \qquad \beta = \frac{\epsilon_2 k_z^{(1)}}{\epsilon_1 k_z^{(2)}}$$

$$\begin{pmatrix} H_2^+ \\ H_2^- \end{pmatrix} = \frac{H_1^-}{2} \begin{pmatrix} 1-\beta \\ 1+\beta \end{pmatrix}$$

$$r_p = \frac{E_r}{E_i} = \frac{E_2^+}{E_2^-} = \frac{H_2^+}{H_2^-} = \frac{1-\beta}{1+\beta} = \frac{n_1^2 k_2 - n_2^2 k_1}{n_1^2 k_2 + n_2^2 k_1} = \frac{n_1^2 n_2 \cos\theta_2 - n_2^2 n_1 \cos\theta_1}{n_1^2 n_2 \cos\theta_2 + n_2^2 n_1 \cos\theta_1} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Hint: Because E_1^- and E_2^- propagate in different media, their ratio is not equal to the corresponding ratio of magnetic fields. Needed correction is obtained from $B = n/cE$ (which holds for PWs).

$$t_p = \frac{E_t}{E_i} = \frac{E_1^-}{E_2^-} = \frac{n_2 H_1^-}{n_1 H_2^-} = \frac{n_2}{n_1} \frac{2H_1^-}{(1+\beta)H_1^-} = \frac{n_2}{n_1} \frac{2n_1^2 k_2}{n_1^2 k_2 + n_2^2 k_1} = \frac{2n_2 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = \frac{2n_i \cos\theta_i}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Thus, the TE and TM material interface transfer matrices are equivalent to imposing the Fresnel equations.

Three-layer structure: Transfer matrix treatment

As an illustration, we derive the expression given in Fowles. It should be evident that more complex structures can be treated the same way.

For simplicity and concreteness, consider the TE polarized case.
This choice does not matter too much:
1. TM and TE results are identical for normal incidence
2. it often sufficient to work with near-normal angles

The transfer matrix representing the three-layer structure is

$$M = M^{(23)} M^{(slab)} M^{(12)}$$

$$M^{(12)} = \frac{1}{2}\begin{pmatrix} 1 + a_{12} & 1 - a_{12} \\ 1 - a_{12} & 1 + a_{12} \end{pmatrix} \qquad M^{(slab)} = \begin{pmatrix} e^{+ik_z^{(2)}l} & 0 \\ 0 & e^{-ik_z^{(2)}l} \end{pmatrix} \qquad M^{(23)} = \frac{1}{2}\begin{pmatrix} 1 + a_{23} & 1 - a_{23} \\ 1 - a_{23} & 1 + a_{23} \end{pmatrix}$$

$$a_{12} = \frac{k_z^{(1)}}{k_z^{(2)}} = \frac{n_1 \cos\theta_1}{n_2 \cos\theta_2} \approx \frac{n_1}{n_2} \qquad a_{23} = \frac{k_z^{(2)}}{k_z^{(3)}} = \frac{n_2 \cos\theta_2}{n_3 \cos\theta_3} \approx \frac{n_1}{n_2}$$

where $\approx$ denotes approximate relations usable for nearly-normal incidence

... matrix multiplication gives

$$M = M^{(23)}M^{(slab)}M^{(12)} = \frac{1}{2}\begin{pmatrix} 1+a_{23} & 1-a_{23} \\ 1-a_{23} & 1+a_{23} \end{pmatrix}\begin{pmatrix} e^{+ik_z^{(2)}l} & 0 \\ 0 & e^{-ik_z^{(2)}l} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1+a_{12} & 1-a_{12} \\ 1-a_{12} & 1+a_{12} \end{pmatrix}$$

$$M = \frac{1}{2}\begin{pmatrix} (1+a_{12}a_{23})\cos(k_z^{(2)}l) + i(a_{12}+a_{23})\sin(k_z^{(2)}l) & (1-a_{12}a_{23})\cos(k_z^{(2)}l) - i(a_{12}-a_{23})\sin(k_z^{(2)}l) \\ (1-a_{12}a_{23})\cos(k_z^{(2)}l) + i(a_{12}-a_{23})\sin(k_z^{(2)}l) & (1+a_{12}a_{23})\cos(k_z^{(2)}l) - i(a_{12}+a_{23})\sin(k_z^{(2)}l) \end{pmatrix}$$

The relation between incident, reflected, and transmitted amplitudes, expressed with this matrix is

$$\begin{pmatrix} E_t \\ 0 \end{pmatrix} = M\begin{pmatrix} E_i \\ E_r \end{pmatrix}$$

The "lower-row" component of this equation reads

$$0 = M_{21}E_i + M_{22}E_r$$

from where we obtain the amplitude reflection coefficient of the tri-layer as:

$$r_s = \frac{E_r}{E_i} = -\frac{M_{21}}{M_{22}}$$

$$r_s = -\frac{(1-a_{12}a_{23})\cos(k_z^{(2)}l) + i(a_{12}-a_{23})\sin(k_z^{(2)}l)}{(1+a_{12}a_{23})\cos(k_z^{(2)}l) - i(a_{12}+a_{23})\sin(k_z^{(2)}l}$$

So one can see that the transfer-matrix method is conceptually simpler than the summation of all possible partial waves. Structures with more layers only add more matrices to the product and everything can be easily programmed.