Additional material: Wave equation as Newton‘s law

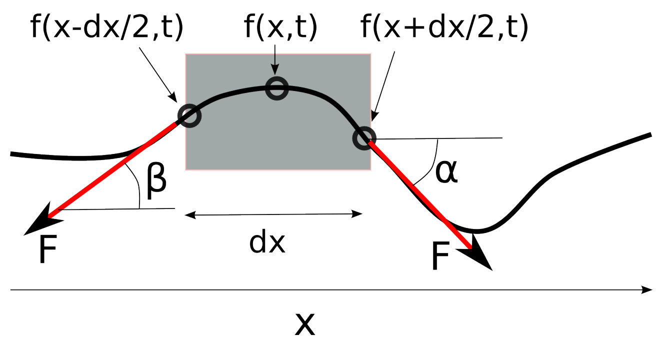
This is to show that the wave equation is merely another way to express equations of motion. We will consider a string, tied between two fixed points, and stretched with a force that we denote F.

Our goal is to apply Newton‘s law to this string and show that it leads to the wave equation derived in the class. We will also determine what is the speed v that will appear in the equation.

First, wee need to characterize our string a bit more precisely. We will consider it to be infinitely thin, so that there is no consideration about what happens in the cross-section of the string. It also means that it is infinitely pliable — it does not require force to bend it.

However, our string must have mass. We will assume that a unit length of this string has mass of ρ. This can be called linear density, and has a physical units of kg per meter.

Now picture a snapshot of the string, and concentrate an its small segment, here shown in the shaded box:



We denote by $f(x,t)$ the *vertical* displacement of the string. We also assume that it is very small. That is also why we can completely neglect the movement in the horizontal direction, along x.

All we need to to is to write Newton‘s law for this small piece of the string:

force equals mass times acceleration

Let us figure out the right-hand side first. The vertical-direction acceleration at the point x is obviously

$$\partial_{tt} f(x,t)$$

Because the mass of the string segment ρdx we have our expression for the right-hand side of what is going to be the wave equation:

$$\text{mass times acceleration} = \rho dx \partial_{tt} f(x,t)$$

Now, determine the force part. The string is tensioned with force F. It pulls in the opposite directions on the ends of the segment, always exactly in the direction that is tangential to the string. Recall that string is pliable, and that is why the force can only be in the direction along the string. This direction is given by the local slope of $f(x,t)$ at any given fixed time t.

The horizontal components of the force cancels (you have to imagine the picture with exceedingly small vertical displacements!). The only relevant is the vertical component of the force:

$$\text{vertical force} = F\sin\alpha - F\sin\beta \approx F(\tan\alpha - \tan\beta)$$

where the last part is because the angles are so small. Finally we recognize that tangents of these angles are nothing but slopes, or derivatives along x, and we get

$$\text{vertical force} = F(\partial_x f(x + dx/2, t) - \partial_x f(x - dx/2, t)) \approx F dx \partial_{xx} f(x, t)$$

where we have used a "definition" of the derivative on the right — this is fine as long as dx is very small, which we can readily assume, because in the end it will cancel. We insert the two pieces into original Newton's equation to obtain:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{F}{\rho} \frac{\partial^2 f(x,t)}{\partial x^2}$$

A comparison with the equation from the class tells us that the string indeed obeys the wave equation, and that the speed that characterizes this equation is given by

$$v^2 = \frac{F}{\rho} \ .$$