P1:

An electromagnetic plane wave propagates along the direction given by a vector $(-1, -1, -1)$. It is linearly polarized, with the direction of oscillation of the electric field given by a vector $(-2, -1, 3)$. This wave is incident on a material interface from air. The medium-air boundary is given by the plane with the equation $y + z = 0$. The refractive index of the material is $\sqrt{2}$.

A Calculate the unit vector $\hat{k}$ in the direction of propagation

Answer:

$$\hat{k} = \frac{1}{\sqrt{3}}(-1, -1, -1,)$$

B Calculate the unit normal $\hat{n}$ of the interface. Orient this vector such that it points into air.

Answer: Coefficients of x, y, and z in the equation of plane give components of a vector that is parallel to the normal:

$$(0, 1, 1)$$

This needs to be normalized to get

$$\hat{n} = \frac{1}{\sqrt{2}}(0, 1, 1)$$

C Determine the plane of incidence. Recall that it is the plane given by the surface normal, and the incident wave propagation direction.

Answer: Plane of incidence is given by two vectors that are parallel to it:

$$\hat{k} \qquad \hat{n}$$

So the normal to the plane if incidence is given by their cross product.

D Calculate $\hat{e}_s$, the unit polarization vector for the s-polarized wave.

Answer: Since the s-polarized wave oscillates in direction of the normal to the plane of incidence, it must be the above-mentioned cross product:

$$\hat{e}_s = \hat{k} \times \hat{n} = \frac{1}{\sqrt{2}}(0, 1, -1)$$

E Calculate $\hat{e}_p$, the unit polarization vector for the p-polarized wave.

Answer: $\hat{e}_p$ must be perpendicular to bot $\hat{e}_s$ and $\hat{k}$. So it is their cross product:

$$\hat{e}_p = \hat{k} \times \hat{e}_s = \frac{1}{\sqrt{6}}(2, -1, -1)$$

F Using Fresnel formulas (e,g in this "symmetric form"),

$$r_\perp = \frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} \qquad r_\parallel = -\frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t}$$

determine the reflection coefficients for both polarizations.

Answer: First one needs to determine the angle of incidence. We can get its cosine from:

$$\cos\Theta_i = |\hat{k}.\hat{n}| = \sqrt{2/3}$$

from where we can get the sine:

$$\sin\Theta_i = \sqrt{1 - \cos^2\Theta_i} = \sqrt{1 - 2/3} = \sqrt{1/3}$$

Next we use law of refraction

$$\sqrt{1/3} = 1 \sin\Theta_i = n_t \sin\Theta_t = \sqrt{2}\sin\Theta_t$$

$$\sin\Theta_t = \sqrt{1/6} \qquad \cos\Theta_t = \sqrt{1-1/6} = \sqrt{5/6}$$

Having both cosines, insert in the Fresnel:

$$r_\perp = \frac{1\sqrt{4/6} - \sqrt{2}\sqrt{5/6}}{1\sqrt{4/6} + sqrt2\sqrt{5/6}} \qquad r_\parallel = -\frac{\sqrt{2}\sqrt{4/6} - 1\sqrt{5/6}}{\sqrt{2}\sqrt{4/6} + 1\sqrt{5/6}}$$

$$r_\perp = \frac{2-\sqrt{10}}{2+\sqrt{10}} \qquad r_\parallel = -\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$$

G Using these values, and relations for reflectance, together with the wave decomposition into s and p polarized components, determine the reflectance for the whole incident wave.

Answer: Because each polarization is reflected independently, the total reflectance is a weighted sum of reflectances for the two polarizations:

$$R = R_s I_s + R_p I_p = I_s r_s^2 + I_p r_p^2$$

where I_s and I_p stand for the fraction of power in the corresponding polarizations:

$$I_s = (\hat{e}_s.\hat{e})^2 = 4/7 \qquad I_p = (\hat{e}_p.\hat{e})^2 = 3/7 \qquad \hat{e} = \sqrt{1/14}(-2,-1,3)$$

As a sanity check, $I_s + I_p = 1$. Finally,

$$R = \frac{4}{7}\left(\frac{2-\sqrt{10}}{2+\sqrt{10}}\right)^2 + \frac{3}{7}\left(\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}\right)^2 = 0.0348$$

Yet another sanity check: This incidence is almost normal, and refractive index is as for "glass" so we expect reflectance of few percent, and that is what we got...

P2:

An electromagnetic plane wave propagates along the direction given by a vector $(0, -1, -1)$. It is linearly polarized, with the direction of oscillation of the electric field given by a vector $(1, -1, 1)$. This wave is incident on a material-air interface from a medium with a refractive index of $\sqrt{2}$. The medium-air boundary is given by the plane with the equation $x + y + z = 0$.

A Calculate the unit vector $\hat{k}$ in the direction of propagation

Answer:

$$\hat{k} = \frac{1}{\sqrt{2}}(0, -1, -1)$$

B Calculate the unit normal $\hat{n}$ of the interface. Orient this vector such that it points into air.

Answer:

$$\hat{n} = -\frac{1}{\sqrt{3}}(1, 1, 1)$$

C Determine the plane of incidence.

Answer: Normal to this plane can be obtained as a cross product: $\hat{k} \times \hat{n}$. Another way is to simply guess a unit vector perpendicular to both of these:

$$\frac{1}{\sqrt{2}}(0, 1, -1)$$

D Calculate $\hat{e}_s$, the unit polarization vector for the s-polarized wave.

Since the TM oscillates along the normal to the plane of incidence, it is just the vector determined above:

$$\hat{e}_s = \frac{1}{\sqrt{2}}(0, 1, -1)$$

E Calculate $\hat{e}_p$, the unit polarization vector for the p-polarized wave.

Answer: This must be perpendicular to both $\hat{k}$ and $\hat{e}_s$, and it is easy to see that must point in the x direction:

$$\hat{e}_p = (1, 0, 0)$$

F Calculate cosines of both the incident and the transmitted angle.

Answer: a) Cosine of the incident is the dot product between the direction of propagation and the normal of the interface:

$$\cos\Theta_i = |\hat{k}.\hat{n}| = \sqrt{2/3}$$

This implies that

$$\sin\Theta_i = \sqrt{1/3}$$

Now apply Snells law:

$$\sqrt{2}\sin\Theta_i = \sin\Theta_t = \sqrt{2/3}\ ,$$

which means that

$$\cos\Theta_t = \sqrt{1/3}$$

G Using Fresnel formulas (write the form you choose to use), determine the transmission coefficients for both polarizations.

Answer:

$$r_s = +\frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} = -\frac{\sqrt{2}\sqrt{2/3} - \sqrt{1/3}}{\sqrt{2}\sqrt{2/3} + \sqrt{1/3}} = \frac{1}{3}$$

$$r_p = -\frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t} = -\frac{1\sqrt{2/3} - \sqrt{2}\sqrt{1/3}}{\dots} = 0$$

$$t_s = \frac{2n_i \cos\Theta_i}{n_i \cos\Theta_i + n_t \cos\Theta_t} = \frac{2\sqrt{2}\sqrt{2/3}}{\sqrt{2}\sqrt{2/3} + 1\sqrt{1/3}} = 4/3$$

$$t_p = \frac{2n_i \cos\Theta_i}{n_t \cos\Theta_i + n_i \cos\Theta_t} = \frac{2\sqrt{2}\sqrt{2/3}}{1\sqrt{2/3} + \sqrt{2}\sqrt{1/3}} = \sqrt{2}$$

It appears that the incidence is Brewster, because the TM reflection coefficient vanishes.

H Determine the percentages of energy in the incident beam that propagates in the s and p polarized waves.

Answer: We have to find the projections of the oscillation direction vector $\hat{e} = \sqrt{1/3}(1, -1, 1)$ on the TM and TE polarization vectors.

$$\hat{e}.\hat{e}_s = \sqrt{1/3}(1, -1, 1).\sqrt{1/2}(0, 1, -1) = -\sqrt{2/3}$$

$$\hat{e}.\hat{e}_p = \sqrt{1/3}(1, -1, 1).(1, 0, 0) = \sqrt{1/3}$$

The squares of these projections are the fractional powers (because power goes with the square of electric field intensity). Thus, fractional of power in the polarization states s, p are (their sum must be one):

$$P_s = 2/3 \qquad P_p = 1/3$$

I Using these values, and relations for transmittance, determine what fraction of power of the whole incident wave is transmitted through the interface.

Answer: If one realizes that the incidence is Brewster, a very simple count can serve as a sanity check:

1/3 of power is transmitted without loss in the TM polarization. Out of 2/3 of total power incident in the TE polarization, one ninth is reflected, and the rest, namely $2/3 * 8/9$ is transmitted. So the total fraction of power transmitted is

$$16/27 + 1/3 = 25/27$$

Now let us calculate this the harder way. Using the formula for transmittance

$$T = t^2 \frac{n_t}{n_i} \frac{\cos\Theta_t}{\cos\Theta_i}$$

with the fractional powers:

$$P = (P_s t_s^2 + P_p t_p^2) \frac{n_t}{n_i} \frac{\cos\Theta_t}{\cos\Theta_i}$$

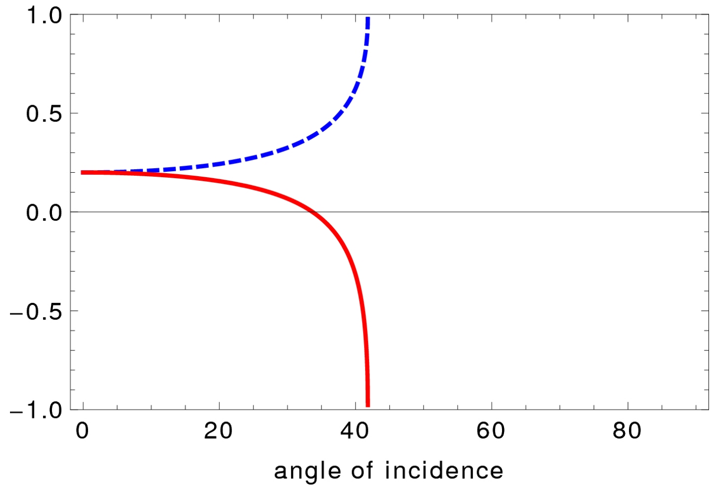
$$P = (2/3 * 16/9 + 1/3 * 2) \frac{1}{\sqrt{2}} \frac{\sqrt{1/3}}{\sqrt{2/3}} = 25/27$$

P3:

Write a Mathematica, Matlab or any other language program to solve the general problem of the same type as the previous two examples.

Assume that three (not necessarily normalized) vectors are given for the direction of propagation, direction of linear polarization (check that the two are orthogonal!), and the normal to the interface. Also, the "incident" and "transmitted" refractive index are known inputs.
Implement both the transmission and reflection quantities. Check energy conservation.

P4:



A Which (Fresnel formulae related) quantities are shown?

Answer: It must be reflection coefficients: The red goes through zero, and only r_p does that. Then they have the same values at zero incidence angle, so the other (dashed) must be r_s.

B Describe three different ways to determine the relative index of refraction for the interface in question.

Answer:

1. From the reflectivity coefficient at $\Theta_i = 0$, which I estimate to be 1/5:

$$\frac{1-n}{1+n} = \frac{1}{5} \qquad n = \frac{4}{6}$$

2. From the critical angle, which I estimate to be $\Theta_c = 42$ deg,

$$n = \sin\Theta_c \qquad n \approx 0.669$$

3. From the Brewster angle, which I estimate to be $\Theta_B = 34$ deg,

$$n = \tan\Theta_B \qquad n =\approx 0.674$$

C Which of the three seems most accurate in this case and why?

Answer: The most accurate is the estimate 1, as the curves seem to start exactly from the value of 2/10.

D Mark the critical and Brewster angles in the picture.

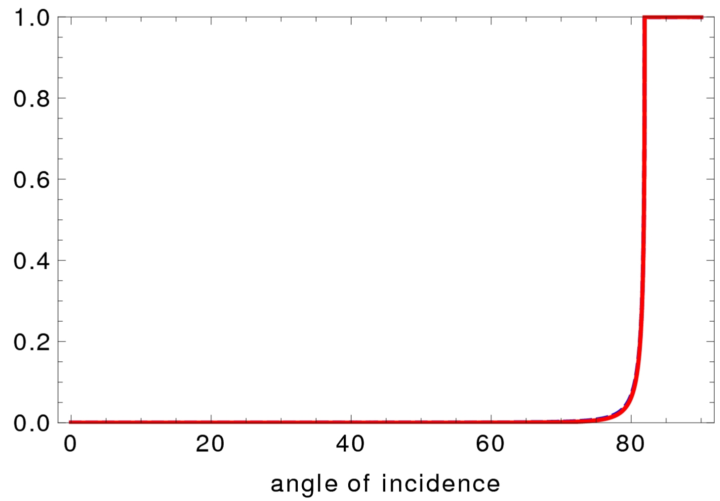
Answer: To find a Brewster angle, I look for z zero crossing (this is the case here), or for unit transmittance or zero reflectance. The latter two *touch* unity and/or zero axis (i.e. no crossing).

E Is this describing internal or external incidence?

Answer: First, we have already estimated relative refractive index to be less than one, $n = n_t/n_i < 1$ so $n_i > n_t$, and this must be internal reflection.

Second way: Look for existence of a critical angle at which the curves in general attain infinite derivatives (vertical asymptote exists).

P5:



A There is two curves almost overlapping in this figure. Which quantities are shown?

Answer: Since the values are between zero and one, they probably represent irradiance reflectance or transmittance. Further, because the values are near zero for normal incidence, they must be reflectances.

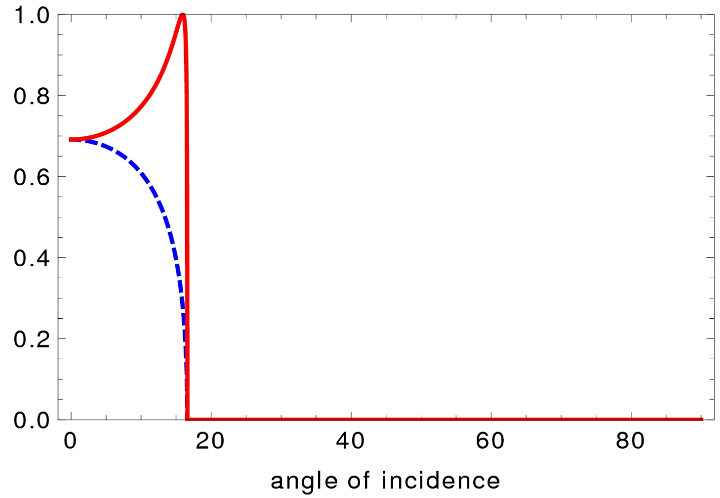
B What can you say about the relative index of refraction? What physical situation it may represent?

Answer: The point at which reflectance reaches unity is obviously the critical angle. It is close to ninety degrees, sine of which will be close to one. Since the sine of critical angle is relative refractive index, the latter must be close to unity.

C Where do you think is the Brewster angle in this picture? Give a qualitative argument supporting your answer.

Answer: We know that tangent of Brewster is the relative refractive index. Together with the previous result, this tells us that this tangent must be close to one and Brewster must be close to 45 deg.

P6:



A Which quantities are shown?

Answer: Quantities go between zero and one, so I guess these are transmittances (for power or intensity, not field amplitude).

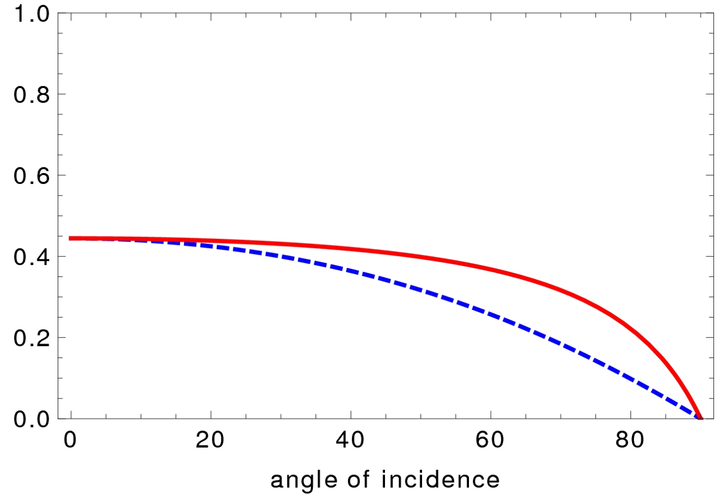
B What can you say about the relative index of refraction? What physical situation it may represent?

Answer: Once again, infinite slope (existence of vertical asymptote) indicates the critical angle. This is at a low value (almost normal incidence) so the index must be high (higher than that of diamond).

C Which curve represents TE polarization?

Answer: Recall that TM (p) polarization has the Brewster. Here it shows as the red curve touching unity. So the dashed must be TE.

P7:



A Is this internal or external incidence?

Answer: There is no signature from a critical angle, so it must be external incidence.

B Are there curves showing reflectivity or transmission coefficients?

Answer: Close to grazing incidence (90 deg), reflectivity would be high, approaching plus/minus one, so these two are transmission coefficients.

C Estimate the relative refractive index.

Answer: In this case, the only option is to use normal incidence:

$$t_p = \frac{2n_i}{n_i + n_t} = \frac{2}{1+n} \approx 0.45 \qquad n \approx 3.4$$

D Evaluate r_s, r_p, t_s, t_p for the incident angle of 60 degrees (or so), and decide which curve belongs to which polarization.

Answer: For 60 deg, we have $\cos\Theta_i = 1/2$ and $\sin\Theta_i = \sqrt{3/4}$, so it is easy to estimate these values. For example

$$t_s = \frac{21/2}{1/2 + \sqrt{n^2 - 3/4}} \approx \frac{21/2}{1/2 + n} \approx 0.25$$

$$t_p = \frac{2n1/2}{n^2 1/2 + \sqrt{n^2 - 3/4}} \approx \frac{1}{n/2 + 1} \approx 0.37$$

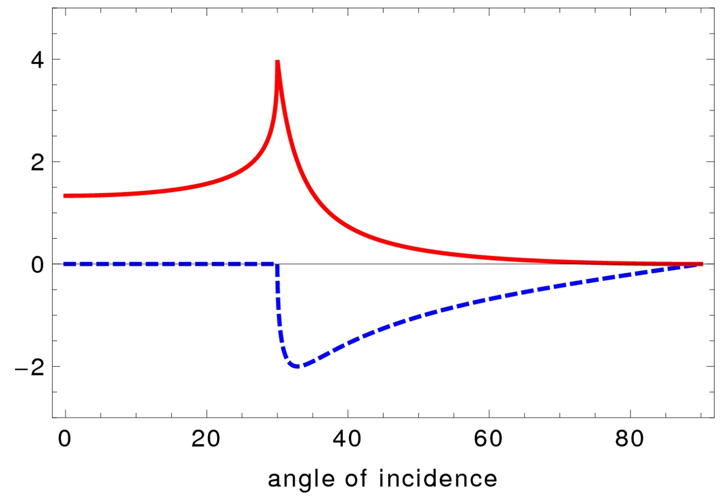
E Extra: give an analytic argument that allows to decide the above question.

Answer: Expand transmission coefficients around $\Theta_i = 0$, and compare the first-order term. You should obtain:

$$t_s \approx t_s(0) - \Theta^2 \frac{(n_t - n_i)n_i}{(n_t + n_i)n_t} \qquad t_p \approx t_s(0) - \Theta^2 \frac{(n_t - n_i)n_i^2}{(n_t + n_i)n_t^2}$$

The second order term in t_p is that in t_p but multiplied by $n_i/n_t < 1$, so this function decreases slower. This implies that the upper curve is t_p.

P8:



This picture shows real and imaginary part of a transmission coefficient.

A Is this internal or external incidence?

Answer: Infinite slopes indicate critical angle, and that means the incidence is internal.

B Which curve represents the imaginary part?

Answer: One does not expect imaginary parts to be around in the "normal" situations, in particular for normal incidence. This suggests that the dashed is the imaginary part. Beyond the critical angle, the square root in the Fresnel becomes imaginary, and this gives rise to nonzero imaginary part at large incidence angles.

C Mark the critical angle.

Answer: The cusp in red curve marks the critical angle.

D Estimate the relative refractive index.

Answer: It seems critical angle is equal to 30 deg. using the formula for critical angle:

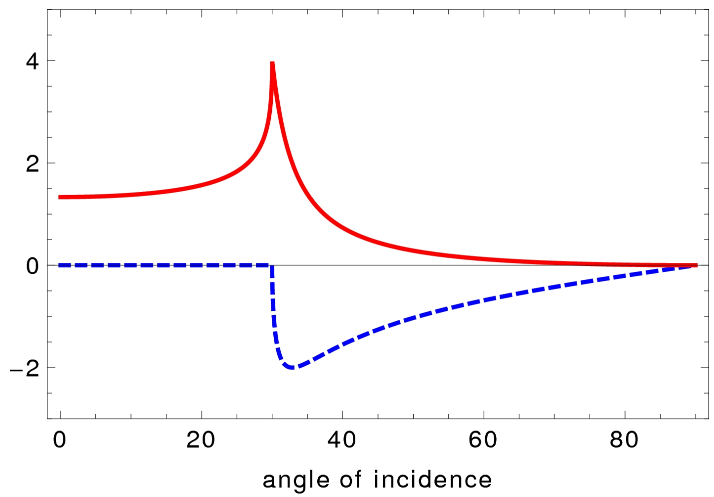
$$n = \sin\Theta_c = \sin(\pi/6) = 1/2$$

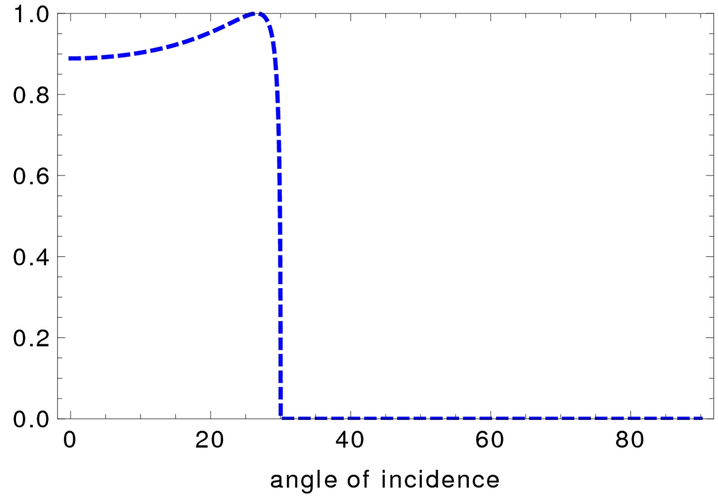
E Show, approximately, where you expect the Brewster angle to be.

Answer: In general before the critical. For a better estimate, use the formula:

$$\tan\Theta_B = n \qquad \Theta_B \approx 26 \text{ deg}$$

P9:





Explain the following "paradox":

The left picture is the same as in the previous example. It shows that a complex valued transmission coefficient exists for large angles beyond the critical. Picture on the right shows the corresponding transmittance, which is zero for $\theta_i > \theta_c$.
We have previously derived

$$T = |t|^2 \frac{n_t}{n_i} \frac{\cos\theta_t}{\cos\theta_i}$$

which seems to suggests that if t is a non-zero, complex-valued quantity for $\theta_i > \theta_c$, T should not be zero. So, strictly speaking there is something wrong with the above formula — what is it? How do you reconcile the two pictures (both correct)?
Answer: The formula for T was derived under assumption that the beam IS propagating in the

second medium - we have used this tacitly when calculation the geometric factor (ration of cosines). However, beyond the critical angle, this is not the case (see notes on Poynting in TIR) and the above formula is not valid for angles larger than critical.

P10:

A)

The four numbers $\{4.84, 2.0, 1.0, -1.0\}$ represent values of reflection and transmission coefficients in some unknown order. Assign these values to r_s, r_p, t_p, t_s. Justify your answer.

Answer: Inspect various plots of reflection and transmission coefficients in the notes. They show that quantities that attain negative values are the reflection coefficients. Further, reflection coefficients are less than one in absolute value. This indicates that the last two are reflection coefficients.

The fact that they are plus/minus one, plus the fact that one other is equal to two implies that we are dealing with the incident angle equal to critical. At the critical angle $t_s = 2$. This tells us that we have to do with the internal incidence, for which it is the r_p that goes to minus one.

Thus, the values are to be assigned as follows:

$$4.84 = t_p \quad 2.0 = t_s \quad 1.0 = r_s \quad -1.0 = r_p$$

Note: The value of t_p at critical angle is also special and related to the refractive index by $t_p = 1/n$ which indicates that the refractive index of the medium in this problem is 2.41.

B)

The same problem for $\{0.46, 0.36, -0.12, -0.64\}$

Answer: These are evidently "random," so the good clue is that we have two of them negative. This occurs for external reflection before Brewster. One can guess that the smallest (in abs) is the p polarized reflection coefficient for incidence angle close to Brewster: $-0.12 = r_p$. The other is then $-0.64 = r_s$. The other two values we assign based on the experience from a previous problem: t_s is the smaller: $0.46 = t_p$, $0.36 = t_s$.