P: This problem pretends to be a "feasibility study" for laser Lunar ranging. Assume that we have a laser producing a single pulse, carrying energyf of $E_p = 1$J at the wavelength of $\lambda = 0.7\mu$m. It is sent toward Moon in the form of a Gaussian beam with the waist $w_0 = 3$m. Our task is to estimate how many photons reach a retroreflector with area $R^2 = 1\text{m}^2$ placed on the Moon surface, and how many will be received back. We will assume that the Earth-Moon distance is $L = 400000$km, and neglect the existence of atmosphere (which simplification makes the result way over-optimistic).

A) Calculate the spot size $w(z = L)$ of the laser beam on the Moon surface. You may approximate the resulting formula by taking into account that the distance is much larger than the beam Rayleigh range.

Answer:

For z much larger than Rayleigh, approximate the square root like so (negleting 1):

$$w(z) = w_0\sqrt{1 + (\frac{z}{z_R})^2} \approx w_0\frac{z}{z_R} = w_0\frac{z\lambda}{\pi w_0^2} = \frac{z\lambda}{\pi w_0}$$

$$w(L) \sim \frac{L\lambda}{\pi w_0}$$

B) Estimate the fraction f of photons that hit the retroreflector - this may be approximated as the ratio between the reflector's and beam-spot areas. Specify f as an expression in terms of given quantities before giving numerical value.

Answer:
This is where we have to work very roughly: Ours will be only *qualitative* estimate, and all numerical factors may be omitted. The idea is to say that the fraction will be proportional to the ratio of two areas:

$$f \approx \frac{\text{area\_reflector}}{\text{area\_spot}} \approx \frac{R^2}{w(L)^2} \approx \frac{R^2\pi^2 w_0^2}{L^2\lambda^2}$$

When you estimate this, you may keep in mind that we have already lost factors like π because of this very approximate way of estimating the fraction — so it is OK to use orders of magnitude to get a feeling how big this quantity may be.

Note: This number of $\sim 10^{-4}$ we get is in fact way too high, because we negleted the fact that the beam gets wider while it travels through the atmosphere... Look e.g. here: http://physics.ucsd.edu/~tmurphy/apollo/basics.html

C) How many photons there are in one laser pulse? What is the energy reaching the retroreflector?

Answer:

$$E_p = N\hbar\omega \qquad N = \frac{E_p\lambda}{\hbar 2\pi c}$$

This will be of the order of 10^{20} or so. Thus, even if a very small fraction will eventually get back, the number of photons detected will be still large...

D) Assuming that the returning beam starts as Gaussian beam with the waist of size R, what is the fraction f' of returning photons that reach the same telescope that launched the beam (so that the "receiver effective radius" is given also by w_0)?

Answer:
This calculation is exactly the same kind as for part B) The only difference is that paremeters w_0 and R exchange their meanings: Now R is the beam waist, and w_0 is the "detector size." So you can say that f' is the same as f but with a change $R \leftrightarrow w_0$. But since f is proportional to both, the exchange does not affect it and we get

$$f' = f$$

So the number of photons coming back will be

$$Nf^2$$

E) True or false? If we were to use laser wavelength $\lambda/2$ instead of λ, but kept the beam power the same, the number of emitted photons would drop by a factor of two. On the other hand, the diffraction angle of the beam would decrease by the same factor (because shorter wavelengths diffract less), thus increasing the fraction of photons on target by factor of two. Consequently, we would have the same number of photons reaching the reflector.

Answer:
The important thing here is keep track of area and not only of linear dimension. In the language of formulas, the number of photons reaching the reflector is

$$N_{target} = Nf = \frac{E_p \lambda}{\hbar 2\pi c} \frac{R^2 \pi^2 w_0^2}{L^2 \lambda^2}$$

So the statement is false because this result still depends on λ.

P: (10pts) This problem deals with the three-dimensional wave equation for scalar waves that propagate with velocity which we denote c. The solution we are going to analyse may represent an approximation for the electromagnetic field inside a corner retro-reflector — in this case the reflecting surfaces of the reflector coincide with the main Cartesian coordinate planes, and we look at a special solution in the first octant, for $x \leq 0$, $y \leq 0$, $z \leq 0$.

A) Write down the correesponding three-dimensional wave equation.

Answer:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

B) Given is a function

$$\psi(x,y,z,t) \equiv \sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(\omega t) \ .$$

Using direct calculation, demonstrate that ψ can be a solution the the wave equation given above, provided a dispersion relation between k_x, k_y, k_z and ω is satisfied. Specify this dispersion relation.

Answer:
Do not do more calculations than really necessary. Here we can't use operator equivalencies directly, because the function is not given in terms of exponentials. Yet, the pattern repeats so it shoudl take no time to get:

$$\frac{\partial^2\psi}{\partial x^2} = -k_x^2\psi$$

and similarly for other coordinates. Either point out this symmetry (so that I know you noticed) or write out explictly everything if you prefer. You should end up with the familiar

$$0 = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - (k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2})\psi = 0$$

Because ψ is in general not zero, you must ask for the factor to vanish:

$$(k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2}) = 0$$

and that is one form of the free-space dispersion relation. So the take-away here should be that even if the wave-form function "looks different" the dispersion relation that one obtains from it must correspond to that found in class.

C) What is the dispersion relation in case the relector is made of glass of refractive index $n = 1.7$?

Answer:
Like we said in the class, when generalizing results from free space to a dielectric medium, find all instances of c and replace them by c/n. You could either make this statement or re-do all calculations to get

$$(k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2 n^2}{c^2}) = 0 \rightarrow |\vec{k}| = \frac{\omega n}{c}$$

D) Function ψ can be characterized as a three-dimensional standing wave which consist of superposition of plane waves. By expressing sin(...) in terms of exp(±...), and by replacing cos(...) by exp(...) give the complex representation of this solution. It is sufficient to specify the solution in a compact, factorized form.

Answer:
This is nothing but a simple use of Euler (for sin) and complex representation (cos → exp)

$$\psi_c = \frac{1}{2i}(e^{+ik_x x} - e^{-ik_x x})\frac{1}{2i}(e^{+ik_y y} - e^{-ik_y y})\frac{1}{2i}(e^{+ik_z z} - e^{-ik_z z})e^{i\omega t}$$

Do not expand... Note that everything before exponential is real-valued so we have as we must:

$$\psi = Re\{\psi_c\} \ .$$

E) Expanding the solution from part D), one could identify each plane wave contributing to ψ. Without doing the calculation explicitly, can you tell how many plane waves there are in ψ?

Answer:
Obviously there will be eight terms, each of them in the form of an exponential. They differ only in the signs in front of wave-vector components, $\pm k_x, \pm k_y, \pm k_z$ — there is eight combinations of those. So our take-away is that each sine or cosine in a standing wave contributes two (counter-propagating) waves but contributions from different coordinate dimensions "multiply."