Practice problems: 1D and 3D waves, complex representation

1. Consider a one-dimensional scalar field $\psi(x,t)$ propagating along the x-axis with velocity v.

(a) Write down the one-dimensional wave equation obeyed by the scalar field $\psi(x,t)$. (2)

$$\left(\partial_{xx} - \frac{1}{v^2}\partial_{tt}\right)\psi(x,t) = 0$$

Or any equivalent form, but check for the opposite sign in front of spatial and temporal derivatives.

(b) Write down a solution for the harmonic wave $\psi(x,t)$ of (temporal) frequency ν, initial phase ε for $x = t = 0$, propagation number k, and amplitude A. (2)

$$\psi(x,t) = A\cos(kx - 2\pi\nu t + \varepsilon) \quad \text{or} \quad \psi(x,t) = A\sin(kx - 2\pi\nu t + \varepsilon)$$

Both choices are OK.

(c) What is the wave velocity v in m/s for a harmonic wave of temporal frequency $\nu = 10^{15}$ s^{-1} and wavelength $\lambda = 0.3$ μm? (2)

$$v = \lambda\nu = 0.3 \times 10^{-6}\text{m} * 10^{15}\text{s}^{-1} = 3 \times 10^{8}\text{ms}^{-1}$$

Never forget units!

(d) For your choice of harmonic wave solution in part (b), and assuming a transverse wave, sketch the spatial variation of $\psi(x,t)$ between $x = 0$ and $x = 2\lambda$ for $A = 1$ and $t = \varepsilon = 0$. (2)

The following things should be part of the sketch:
- exactly two periods of cosine- or sine-like function
- the graph should start with zero or unity at the origin, depending on your choice (sin vs cos) of the harmonic solution
- label your axes
- mark the wavelength on the horizontal axis
- mark the amplitude on the vertical axis

2. This problem involves the complex plane-wave solution $\psi(\vec{r},t) = A\exp[i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)]$ of the three-dimensional wave equation. Assume that ε was chosen such that A is real-valued.

(a) Explain how one extracts the physical solution from the above complex plane-wave solution. (2)

$$\text{Physical\_Field} = Re\{\psi(x,t)\}$$

(b) Based on your answer from part (a) write down the physical harmonic solution associated with the complex plane-wave solution. (2)

$$\text{Physical\_Field} = Re\{\psi(x,t)\} = Re\{A\exp[i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)]\} = A\cos(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

Note: This works when A is real-valued. Then simply replace exp by cos and remove i from the argument of the exponential. However, sometimes A may also be complex-valued. For example we may have

$$\psi(\vec{r},t) = A\exp[i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon) = |A|e^{i\phi_A}\exp[i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

In this case you need to add the phase of A, i.e. $i\phi_A$ to the argument of the exponential and only then replace exp by cos, like so:

$$\text{Physical\_Field} = Re\{|A|e^{i\phi_A}\exp[i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)\} = |A|Re\{\exp[i\phi_A + i(\vec{k}\cdot\vec{r} - \omega t + \varepsilon)\} = |A|\cos(\phi_A + \vec{k}\cdot\vec{r} - \omega t + \varepsilon)$$

In other words, phase of a complex-valued A contributes to the phase of the physical solution.

(c) What is the name and symbol for the quantity that determines the direction of propagation of the plane-wave solution? (2) Specify it for a case of a wave propagating at 45 deg with respect to the z-axis and at 90 deg w.r.t. the x-axis. Assume that the wave propagates in the vacuum and has the wavelength of 500nm.

$\vec{k}$, wave vector, here written such that a unit direction vector is "factored out:"

$$\vec{k} = \frac{2\pi}{\lambda}(0,1,1)/\sqrt{2} = 1.26\times 10^7\frac{(0,1,1)}{\sqrt{2}}m^{-1}$$

(d) For $t = 0$ consider a surface which is transverse to the propagation vector $\vec{k}$ of the complex plane-wave solution. What can you say about the value of the complex plane-wave over such a surface? Specify one example of such a plane. (2)

... that it is constant. A simple example is a plane that goes through the origin of the coordinate system, here given by the equation:

$$y + z = 0$$

In general, when you are given a wave-vector specified by, say, its components k_x, k_y, k_z, the planes that are perpendicular to $\vec{k}$ are given by equations in the form

$$a_x x + a_y y + a_z z = const$$

where numbers $a_{x,y,z}$ are proportional to $k_{x,y,z}$. Note that the planes we are referring to in this way are the *wavefronts.*

(e) Consider two solutions ψ_1 and ψ_2 of the linear three-dimensional wave equation. Circle the combinations of the two solutions below that would be legitimate solutions ψ of the wave equation: (2)

$\psi = (\psi_1 + \psi_2), \quad \psi = \psi_1 \psi_2, \qquad\qquad \psi = \frac{1}{2}(\psi_1 - \psi_2)$

$\psi = \psi_1/\psi_2, \qquad \psi = \frac{1}{100}(\psi_1 + i\psi_2), \quad \psi = \psi_1^2$

Answer: The valid solutions are, in the typewriter order, the first, third, and fifth. Make note of the fifth in particular — you CAN combine different solutions with complex coefficients.

Note: This is a poorly worded question: There are "pathological examples" when the other cases might become solutions to wave equations. For example if you take $\psi_2 \equiv 0$, then also the second, $\psi = \psi_1 \psi_2 \equiv 0$ is a solution, albeit trivial.