P1: (10pts)
This problem is about general one-dimensional wave equation, and its solutions.

A) (2pts) Write down the wave equation for propagation in one spatial dimension, and for the propagation speed v. Denote the spatial and temporal variables z and t, respectively.

$$\partial_{zz}\psi - \frac{1}{v^2}\partial_{tt}\psi = 0$$

B) (3pts) Show that

$$\psi = A\sin\left[\cos\left(\sin[\cos(z-vt)]\right)\right]$$

is an exact solution to the wave equation you gave in problem A).

Answer: This has a form of the general solution to 1D WE, $V(z-vt)+U(z+vt)$ with $U=0$ and $V(.) = \sin\left[\cos\left(\sin[.]\right)\right]$, ψ thus must be a solution.

C) (2pts) Write down a complex representation for a harmonic solution with wavelength λ, and say how is the physical solution obtained from its complex representation. wave?

$$\hat{\psi} = A\exp[i\frac{2\pi}{\lambda}(z-vt)] \quad , \quad \psi = \mathrm{Re}\{\hat{\psi}\}$$

D) (2pts) Which of these IS NOT a solution to the wave equation you gave in problem A)? Justify your answer.

1. $\exp(z+vt)$
2. $\cos(a(z-vt))$ with a standing for a positive real constant
3. $(z-vt)/(z+vt)$
4. $\sin(z+vt) * \cos(z+vt)$
5. $\sin(z-vt)/(z-vt)$

Hint: To avoid lengthy calculations, it helps to recall the general form of the solution, and identify the admissible cases first.

Answer: 3. Because only case 3. does not have a form of a general solution (see B).

P2: (10pts)
This problem is about vector properties of plane wave solutions to Maxwell equations. Consider a plane wave in vacuum given by the following:

$$\vec{E} = \hat{e} E_0 \cos(\vec{k}.\vec{r} - \omega t) \ ,$$

where the polarization-direction unit vector $\hat{e} = (\hat{\imath} + \hat{\jmath})/\sqrt{2}$, and the wave-vector $\vec{k} = |\vec{k}|(\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$. The wavelength of this plane wave is $\lambda = 400$ nm, and its irradiance is $I = 1\ W/m^2$.

A) (1pts) What is the angular frequency ω of this wave?

$$\omega = \frac{2\pi c}{\lambda} = \frac{2\pi\ 3\times 10^8 \mathrm{m/s}}{400\times 10^{-9}\mathrm{m}} = \frac{3\pi}{2} 10^{15} \mathrm{s}^{-1}$$

B) (1pts) Using dispersion relation for plane waves in vacuum, determine the magnitude $k = |\vec{k}|$ of the wave-vector $\vec{k}$.

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \frac{2\pi}{400\times 10^{-9}\mathrm{m}} = \frac{\pi}{2} 10^7 \mathrm{m}^{-1}$$

C) (3pts) What is the value of E_0? Find a formula for E_0 from the magnitude of the time-averaged Stokes vector, and make sure you show all physical units when you evaluate magnitude and physical units of E_0. Hint: In dimensional calculations, it is often convenient to express unit of power, Watt, as product of Ampere and Volt, W = V A.

$$I = \langle \vec{S} \rangle_T = \frac{1}{2}\epsilon_0 c E_0^2 \quad \rightarrow \quad E_0 = \sqrt{\frac{2I}{c\epsilon_0}}$$

$$E_0 = \sqrt{\frac{2VA/m^2}{3\times 10^8 m/s\ 8.854\times 10^{-12} As/(Vm)}} \approx 27.4\frac{V}{m}$$

D) (3pts) Calculate the polarization-direction unit vector $\bar{b}$ for the magnetic field of this plane wave. Say in which order the three unit vectors $\hat{b}$, $\hat{e}$, and $\vec{k}/|k|$ constitute the right-hand oriented system and use that knowledge to calculate $\hat{b}$.

$$\hat{b} = \frac{1}{|k|}\vec{k}\times\hat{e} = (\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3} \times (\hat{\imath} + \hat{\jmath})/\sqrt{2} = \frac{1}{\sqrt{6}}(2\hat{k} + \hat{\jmath} - \hat{\imath})$$

E) (2pts) Write down the formula for the magnetic field in terms of $\hat{b}$ and E_0, and evaluate (including physical SI unit!) the amplitude of the magnetic field B_0.

$$\vec{B} = \hat{b}\frac{E_0}{c}\cos(\vec{k}.\vec{r} - \omega t)$$

$$B_0 = E_0/c \qquad [B_0] = \frac{Vs}{m^2}$$

P3: (10pts)
This problem concerns Maxwell equations in the differential form, and properties of electromagnetic plane waves. Using Faraday's law, you will determine the relation which must be satisfied between the amplitudes of the electric and magnetic field in the electromagnetic plane wave which propagates through a medium with refractive index n, and is described by the following electric and magnetic fields,

$$\vec{E} = \hat{j} E_0 \sin[(2\pi n z/\lambda + \omega t)] \qquad \vec{B} = \hat{\imath} B_0 \sin[(2\pi n z/\lambda + \omega t)] \ ,$$

where λ stands for the *vacuum* wavelength.

A) (2pts) Write down the dispersion relation for electromagnetic waves in the dielectric medium with refractive index n. Express the dispersion relation both in terms of the propagation number k and also in terms of the vacuum wavelength λ

$$k = \frac{\omega n}{c} \qquad \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

B) (2pts) The above fields imply the wave-vector $\vec{k}$ of the plane wave. Write its components in terms of parameters given, and say in which directions this plane wave propagates (making sure to distinguish propagation along positive and negative axis)

direction: negative z axis due to positive relative sign in the phase argument

$$\vec{k} = -\frac{2\pi n}{\lambda}\hat{k} \quad \text{or} \quad \vec{k} = (0, 0, -2\pi n/\lambda)$$

C) (2pts) Write the Faraday's law for the case of dielectric (non-magnetic) medium with refractive index n. Express this vectorial equation in terms of the refractive index and in terms of the magnetic field $\vec{B}$ and electric field $\vec{E}$, in the differential form.

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

D) (3pts) Write the above equation in the component form, showing all three of its components

$$\begin{aligned} \partial_y E_z - \partial_z E_y &= -\partial_t B_x \\ \partial_z E_x - \partial_x E_z &= -\partial_t B_y \\ \partial_x E_y - \partial_y E_x &= -\partial_t B_z \end{aligned}$$

E) (2pts) Evaluate the equations obtained in the previous task for the given electric and magnetic fields

Only the x component of the above is non-trivial:

$$-\frac{2\pi n}{\lambda} E_0 \cos[2\pi n z/\lambda + \omega t] = -B_0 \omega \cos[2\pi n z/\lambda + \omega t]$$

F) (1pts) Using the dispersion relation from problem A), find the relation between the electric and magnetic amplitudes E_0 andB_0.

$$\frac{n}{c} E_0 = B_0$$

P4: (5pts)
This problem deals with quantum nature of light, and in particular with photons and their properties. Femtosecond laser pulse, propagating in vacuum, has the angular frequency $\omega = 7.0 \times 10^{15}\text{s}^{-1}$, and total energy of $E = 10$ mJ. The waist of the laser beam is $w_0 = 1$ cm.
A) (1pt) What is the energy of a single photon with this angular frequency?

$$E_{ph} = \hbar\omega = 1.0 \times 10^{-34}\text{Js } 7.0 \times 10^{15}\text{s}^{-1} \approx 7 \times 10^{-19}\text{J}$$

B) (1pt) What is the momentum of a single photon in this pulse?

$$p_{ph} = \hbar k = \hbar\frac{\omega}{c} = \frac{E_{ph}}{c} = \frac{7}{3} \times 10^{-27}\text{kg m/s}$$

C) (1pt) What is the number N_{ph} of photons contained in the pulse?

$$N_{ph} = \frac{E}{E_{ph}} = \frac{E}{\hbar\omega} = \frac{10^{-2}\text{J}}{7 \times 10^{-19}\text{J}} \approx 0.14 \times 10^{17}$$

D) (1pt) What is the total momentum of this laser pulse?

$$p = N_{ph}\hbar k = \frac{E}{\hbar\omega}\hbar\frac{\omega}{c} = \frac{E}{c}$$

E) (1pt) An "absolutely black" particle with a mass of $M = 10^{-6}$ kg, initially at rest, completely absorbs the laser pulse. Based on conservation of momentum, calculate its resulting velocity, and indicate in which direction will the particle move? Discuss why the result doe not depend on the value of Planck constant.

Conservation of momentum: Particle momentum equals total pulse momentum.

$$Mv = p = \frac{E}{c} \qquad v = \frac{E}{Mc} = \frac{10 \times 10^{-3}\text{J}}{10^{-6}\text{kg } 3 \times 10^{8}\text{m/s}} = \frac{1}{3}10^{-4}\text{m/s}$$

The reason that the result does not depend on Planck is that the various force effects of the el-mag field do exist also in the non-quantum regimes, and it must therefore by possible to describe them without quantum notions. So even if we talk "quantum," references to Planck must cancell out in the end.
Note 1. that this is for absorbtion. If the pulse was reflected, the result would be twice as big, because pulse momentum change would be equal twice its original value.
Note 2. You must not use energy conservation to evaluate velocity. This is because you have no guarantee that the pulse energy is all converted to kinetic energy.

P5: (5pts)
A Gaussian laser beam has power of $P = 2$W, and the beam size w_0 is such that the Rayleigh range of the beam is $z_R = 10$ m. The wavelength is $\lambda = 1\mu$m.

A) (1pts) Write down the expression for z_R in terms of λ and w_0. Explain the physical meaning of Rayleigh range.

$$z_R = \frac{\pi w_0^2}{\lambda}$$

One possible answer: Distance at which intensity decreases to one half of maximum.
Another: Distance at which beam size increases by a factor of $\sqrt{2}$.
B) (1pts) Write down the expression for the beam cross-section A in terms of w_0.

$$A = \frac{\pi w_0^2}{2}$$

C) (1pts) Calculate the maximal intensity or irradiance in the beam.

$$I = \frac{P}{A} = \frac{2P}{\pi w_0^2} = \frac{2P}{\lambda z_R} = \frac{4\text{W}}{10^{-6}\text{m } 10\text{m}} = 4 \times 10^5 \text{W/m}^2$$

D) (2pts) How many photons per second does this beam carry?

Answer: Power divided by energy of a single photon.

$$\Phi_{ph} = \frac{P}{\hbar\omega} = \frac{P\lambda}{\hbar 2\pi c} \approx 10^{19}\text{s}^{-1}$$

Sanity check: It is good to know that 1μm photn is about 1.24 eV energy which in turn is of the order of 10^{-19}J. Thus if we have of the order of Watts in power in this case, the results shoud be about $1/10^{-19}$.