Final: Practice II, with brief solutions

P:

This problem deals with the Fabry-Perot interferometer. We have derived the following formula formula for its transmissivity

$$I/I_0 = \frac{T^2}{(1-R)^2}\frac{1}{1+F\sin^2(\Delta/2)}$$

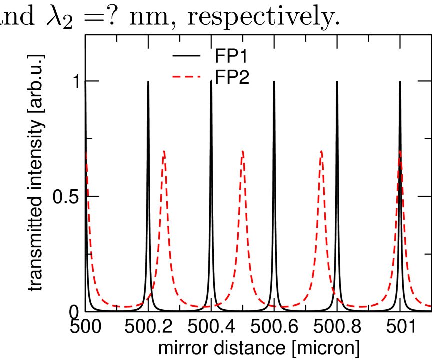
where $\Delta = 2kd\cos\theta$ (the medium between mirrors is air, and we neglect additional phase change δ_r from the reflection on the mirrors).

A) Assume that the mirror distance is 0.5 mm, reflectance of each mirror is equal to 0.8, and transmittance is 0.2. Make a qualitative sketch of transmitted intensity versus wavelength (for, say, normal incidence) and:

1. Mark the points that correspond to transmission resonance
2. Identify the free spectral range, evaluate its value in units of frequency (i.e. 1/s)
3. Indicate in your sketch the quantities that determine the finesse
4. Calculate the minimal value of transmittance, and indicate in your sketch the corresponding points.

The sketch should show a sequence of (roughly) regularly separated peaks, with flat valleys in between. Maximal peak values should be equal to one because this (ideal) FP has no loss. These are the points that correspond to resonance. Spectral range is the separation between peaks (irrespectively of what units are on horizontal axis). In units of per second, free spectral range as $c/2d$, which in turn is the inverse round-trip time in the FP cavity. Two quantities determine the finesse: It is the ratio of the full-width-at-half-maximum of the peak to the distance between peaks. Minimal transmittance occurs when sin in the FP formula become equal to one. With no losses the prefactor $T^2/(1-R)^2$ is equal to one, and the minimal transmittance is therefore $1/(1+F)$. The value of the coefficient of finesse F should be about 80 for for $R = 0.8$, so about 1/81 part of the intensity is transmitted off resonance.

B) The following figure shows the transmittance of two different FPs, labeled FP1 and FP2, as functions of the mirror distance (which is the same in both FP1 and FP2), when illuminated by light with wavelengths of $\lambda_1 = 400$ nm and $\lambda_2 =?$ nm, respectively.



1. Which of the interferometers has mirrors with higher reflectance?

R controls the width of the resonance peaks. So the easiest way to answer this is to compare the width of these peaks: FP1 has higher R.

Alternatively, one could also compare the minimal transmission. It is evidently higher for FP2, so we can conclude that FP2 has lower R

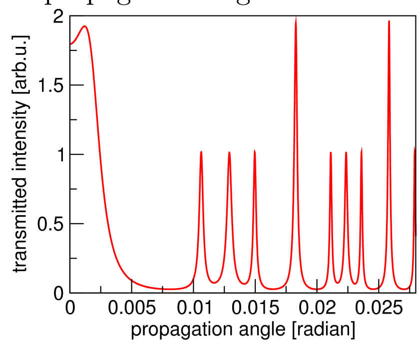
2. Which FP is ideal (has no losses) and which has mirrors with some absorption? Based on the graph, estimate the absorption A of the lossy mirror if you know that its reflectance is 0.7.

For this, compare the max peak values: The decrease from one is controlled by A. Obviously FP2 has lossy mirrors.

3. What is the other wavelength?

Look for peaks that coincide. One coincidence occurs for mirror distance $d = 500\mu$m, the next is at 501μm. Count spaces between: you get 5 and 4 for FP2 and FP1, respectively. This says: to 5 and 4 wavelengths are needed to cover the additional mirror round-trip distance. So the ratio of wavelength is 5/4, and thus the other wavelength is 500 micron.

C) A FP is illuminated by a source with two discrete wavelengths λ_1 and λ_2. The transmitted intensity is shown here as a function of the propagation angle:



1. What is the fraction of the incident light power carried by each wavelength? Justify.

We have only two heights of peaks. The higher one obviously correspond to overlapping signal from the two wavelengths. The lower ones are either shorter or longer wavelength, but because all of the lower peaks are equal, the intensity carried by each wavelength must be equal.

2. If the longer wavelength is 500 nm. What is the other wavelength?

Here we have to guess which peaks correspond to which wavelength. For zero angle the transmission appears to be close to maximal, so we must have both wavelength near resonance for zero angle. This central peak is there for S+L (indicating shorter and longer wavelength contributing). The next peak must be S (indicating that only the shorter wavelength contributes). The high peaks must be also S+L. So the sequence is:

$$S+L \quad S \quad L \quad S \quad S+L \qquad S \quad L \quad S \quad S+L \quad \ldots$$

As I said above, since the peak at $\theta \approx 0$ is as strong as other strong peaks, so we must have a near-resonance for small angle and *both* wavelengths. This is therefore the first coinciding pair of peaks. The next is the first $S+L$ peak. For this we have to count "spaces" for both S and L separately. We find that we need 3 and 2 wavelength to cover additional distance at the higher angle of propagation. So the ratio of wavelength is 2/3, which gives 333nm for the other wavelength.

3. Propose a method to estimate the FP mirror distance from the information in the above graph. Hint: Note the behavior of transmitted intensity in the vicinity of $\theta \approx 0$. Make an argument that we have a near resonance for zero angle. Choose an appropriate peak position to relate the mirror distance and wavelength to the corresponding angle. Use the fact that angles are small.

The feature (small dip) at $\theta = 0$ indicates that the peaks "just starts to move away from $\theta = 0$. Here we assume that is has not happened yet, and $\theta = 0$ is still at resonance. We need one more peak for which we can read its angle accurately. There is one at $\theta_x = 0.015$, and it corresponds to the shorter wavelength.

We use this peak in the FP formula, specifically in the argument of the sin function. It must give zero because we are at resonance at θ_x:

$$\sin\left(kd\cos(\theta_x)\right)^2 = 0$$

Now we use the fact that the angle is small:

$$\sin\left(kd\cos(\theta_x)\right)^2 \approx \sin\left(kd(1-\theta_x^2/2)\right)^2 = 0$$

Next we use the fact that because $\theta = 0$ is also at resonance, and therefore

$$\sin{(kd)}^2 = 0$$

This means that kd is a multiple of π, which in turn implies

$$\sin\left(kd(1-\theta_x^2/1)\right)^2 = \sin\left(kd\theta_x^2/2\right)^2 = 0$$

Form this we know that

$$kd\theta_x^2/2 = 2\pi$$

The factor 2 appears because θ_x corresponds to the *second* resonance peak counted from zero angle. So finally we have

$$d = \frac{2\lambda}{\theta_x^2} \approx 2.962\ mm$$

which is not too far from 3mm used to generate this figure. The inaccuracy is mainly due to the fact that $\theta = 0$ is not exactly at resonance.

P: This problem concerns properties of optical coatings. A substrate of glass with the refractive index $n_s = 1.7$ is coated with a layer of transparent material with the index $n_c = 1.3$.

A) Is this layer designed as a high reflectivity or anti-reflective coating?

Recall that for AR coatings the index of the top layer must be between the index of the medium "above" (which is silently assumed to be air in this case) and the index of the medium "below" i.e. the substrate. You may also recall that the ideal refractive index value is the geometric mean $\sqrt{n_a n_s} = \sqrt{n_s}$. The value $n_c = 1.3$ is not too far from it so this must be an AR coating.

B) What should be the thickness h of the coating if it is supposed to work best at the wavelength of $\lambda = 520$ nm?

The thickness of the layer is "always"one quarter of the wavelength *in the material.* for the given values

$$h = \frac{\lambda}{4n_c} = 100\text{nm}$$

C) What should h be if it is designed for the incidence angle of 45 degrees?

What counts toward the one-quarter wavelength criterion is actually the longitudinal component of the in-medium wavevector. [Recall that it is this quantity that appears in the transfer matrices, and all angular dependencies are realized through k_z.]

So to find k_z, in the coating medium the dispersion relation is

$$k_z^2 + k_{||}^2 = k^2 n_c^2$$

The second term is the same as in the air above (because this component is along the interface and as such it is conserved), namely

$$k_{||} = k \sin\theta_i = k/\sqrt{2}$$

So we obtain

$$k_z^2 = k^2(n^2 - 1/2) \ .$$

Having found k_z (we could equally well use the Snell's law, of course) we can now formulate the requirement that the layer is quarter wavelength w.r.t. the k_z:

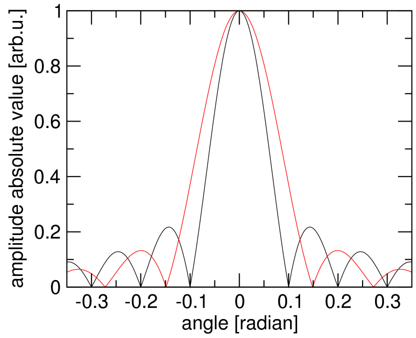
$$k_z d = \pi/2$$

from where one gets

$$d = \frac{\lambda}{4\sqrt{n^2 - 1/2}}$$

So the layer must be thinner. Sanity check: This formula goes to the usual $\lambda/(4n)$ for normal incidence as it must.

P: This problem deals with Fraunhofer diffraction. The picture shows the absolute value of the field amplitude (to emphasize low-intensity features) versus diffraction angle. One of the curves represents diffraction on a slit, and the other diffraction on a circular aperture.



A) Identify which curve represents diffraction on a slit and which corresponds to a circular aperture. Justify your answer.

One needs to examine the special points in both graphs. While the maxima occur at "non-friendly" values, minima of intensity correspond to zeros of either sin in case of the slit, or zeros of the sombrero function in case of a circular aperture. So all we need to recall is that zeros of sin are equidistantly spaced. Only one of the curves, the black one, has this property, so it must represent diffraction on the slit.

B) What is the width of the slit, expressed in units of wavelength of the incident light?

Here we use the formula

$$I \sim \frac{\sin\beta}{\beta} \qquad \beta = \frac{kb}{2}\theta$$

and identify the first zero at $\theta = 0.1$ which must occur at the value

$$\pi = \beta_x = \frac{kb}{2}0.1$$

which gives $b = 10\lambda$. This was indeed the value used to generate the curve.

C) Based on the previous result (and the figure, of course), give an estimate of the circular aperture diameter.

A qualitative answer is sufficient here. Because the angular extent of the central lobe is comparable in the two curves, also comparable must be the characteristic dimensions of the slit (total width b) and of the circular aperture (diameter). So we can estimate the diameter to be not far from 10λ. It was 8.2λ when I generated this picture.

A more precise estimate can be obtained from the formula for the Airy disk... For this you have to estimate location of the first zero in the red curve, plus recall the Airy disk formula e.g. in the form

$$q1 = 1.22\frac{L\lambda}{2a}$$

where l is the observation distance and $2a$ is the diameter to be determined. Expressed in terms of the propagation angle,

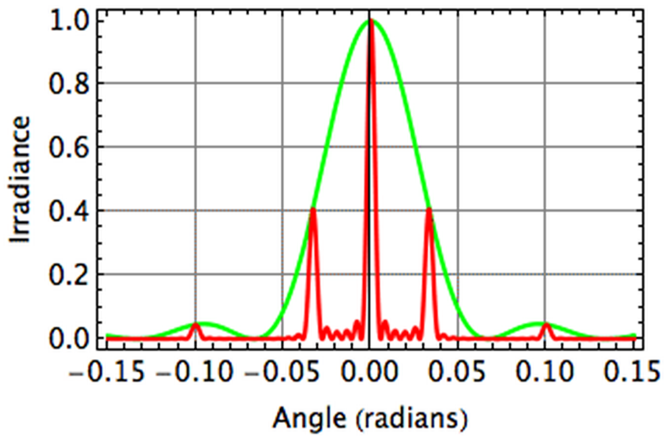
$$\theta_1 = 1.22\frac{\lambda}{2a}$$

If we roughly estimate (from the figure) $\theta_1 = 0.15$, then the diameter guess becomes

$$2a = 1.22 \frac{\lambda}{0.15} \approx 8.13\lambda$$

Obviously this is much more quantitative estimate...

P: N-slit as a diffraction grating. Consider Fraunhofer diffraction on an aperture consisting of N equivalent slits, characterized by their widths a and their center-to-center distance d.
The following figure shows the intensity pattern in the far field.



A) What is the meaning of the green-curve "envelope" in this figure?

The green curve is the "envelope" given by the diffraction properties of the single slit. Single slit is the finest (smallest) feature of the N-slit, and as such it contributes the "angularly widest" feature of the far field pattern.

B) There are three characteristic angle scales in this figure. Identify them in the graph and say to which parameter of the N-slit are they related.

The first scale we have already identified: it is given by the green curve. We can take for it the distance form the center to the first green zero.

The second largest angular scale is the distance between the central peak and the first strong peak. This is influenced by the second smallest feature of N-slit which is the slit-to-slit distance d.

The third scale is the finest - it is the distance between subsidiary (small) peaks. This is determined by the longest scale of the N-slit which is it s"total size" namely Nd.

C) Extract the necessary information from this picture and determine as much as you can about the parameters (N, a, d) of this N-slit.

Formula to use:

$$I \text{sinc}\beta \ \text{sinc}\alpha \qquad \beta = \frac{kb}{2}\theta \qquad \alpha = \frac{kd}{2}\theta$$

In which we insert θ values corresponding to first zeros to get

$$\frac{kb}{2}0.07 = \pi \qquad \frac{kd}{2}0.035 = \pi$$

giving

$$b = 14.2\lambda \qquad d = 28.5\lambda$$

The actual values used in the applet that generated the figure were 15 and 30. The inaccuracy is obviously in estimating the zeros from the figure.

The remaining parameter is N. To get this we recall that N divides the second scale into N sections. So we start counting in the central peak and count maxima until we reach the second strongest peak. We get $N = 6$.

D) Imagine that this N-slit is irradiated by a light beam that contains two colors, say red and blue. Sketch the qualitative picture of the resulting intensity pattern.

It would be a “sum” of two red curves from the above figure, each having a different horizontal scale. The red color would have a secondary peak at larger angle: This is because longer wavelength bend and diffract more.

E) Describe in simple qualitative terms what kind of pattern would you see (with a color-sensitive detector such as human eye) if the N-slit is illuminated by white light.

The central peak would be more or less white, as all colors overlap in it. The secondary peak would turn into bands of colors. Reds would be on the outer “edges.”

P: A high-power femtosecond laser pulse has a peak intensity (irradiance) of $I_0 = 10^{15}$ W/m^2, and a Gaussian beam waits of $w_0 = 1$ cm. The duration of the pulse is $\tau = 50$ fs, and the central wavelength is $\lambda = 780$ nm.

A) What is a single photon energy corresponding to this pulse wavelength?

B) What is a single photon momentum p corresponding to this pulse wavelength?

$$p = \hbar k = \hbar \frac{2\pi}{\lambda} = \frac{h}{\lambda}$$

C) Calculate the peak pulse power P in Watts.

You need to recall the formula for the area of a Gaussian beam

$$A = \pi w_0^2/2$$

Then

$$P = IA$$

D) Estimate the pulse total energy E_p in Jules.

$$E_p \sim P\tau$$

E) Estimate the number of photons in the pulse.

Use the single-photon energy $\hbar\omega$:

$$N_{ph}\hbar\omega = E_p = P\tau = \pi w_0^2 I\tau/2$$

so

$$N_{ph} = \frac{\pi w_0^2 I\tau}{2\hbar\omega} = \frac{\pi w_0^2 I\tau\lambda}{2\hbar 2\pi c} = \frac{w_0^2 I\tau\lambda}{4\hbar c}$$

F) The pulse is completely absorbed on a non-reflecting target. Estimate the total momentum force Δp transferred on the target.

$$\Delta p = N_{ph} p = \frac{w_0^2 I\tau\lambda}{4\hbar c}\hbar 2\pi/\lambda = \frac{\pi w_0^2 I\tau}{2c}$$

G) Use the previous result to estimate the peak force exerted on the target.

$$F\tau = \Delta p \qquad F = \frac{\pi w_0^2 I}{2c}$$

This gives a force of the order of kN which is quite large. But it acts for a very short time...