Problem 1:
Consider the plane wave traveling in free space with the wave vector $\vec{k} = 2\pi(1,1,1)/(\sqrt{3}\lambda)$, where $\lambda = 600\text{nm}$, and with an electric field amplitude of $E_0 = 30\text{V/m}$. The phase of the wave is 2π at the origin (i.e. the magnitude of the electric field vector at $x = 0, t = 0$ is E_0). Direction of the electric field oscillation is given by a unit vector $\hat{e}$.

A) Find an expression for the wave using a real representation. State the condition for $\hat{e}$ that must be satisfied to ensure that divergence of the electric field vanishes.

$$\hat{e}E_0 \cos[\vec{k}.\vec{r} - \omega t] \quad \text{where} \quad \omega = \frac{2\pi c}{\lambda} \quad \text{and} \quad \hat{e}.\vec{k} = 0$$

B) Find an expression for the wave using the complex notation.

$$\hat{e}E_0 \exp[i(\vec{k}.\vec{r} - \omega t)]$$

C) Find the frequency ν of the wave in THz.

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8 ms^{-1}}{6 \times 10^{-7} m} = 0.5 \times 10^{15} s^{-1}$$

D) Find a point other than the origin where the phase is 2π at $t = 0$

$$\text{any point for which} \quad \vec{k}.\vec{r} = 2\pi \quad \text{for example} \quad \vec{r} = (\sqrt{3}\lambda, 0, 0)$$

E) Describe (either using words or by giving an equation) the phase fronts of this wave

1. phase front = any plane that is perpendicular to $\vec{k}$
2. $\vec{k}.\vec{r} = \text{const}$

F) Give two possible choices for $\hat{e}$ (so that divergence of the field is zero)

$$\hat{e}_1 = (1, -1, 0)/\sqrt{2} \qquad \hat{e}_2 = (1, 1, -2)/\sqrt{6}$$

G) For your first choice in F), specify the unit vector $\hat{b}$ for the direction of oscillation of the magnetic field

$$\hat{b} = (-1, -1, +2)/\sqrt{6} = -\hat{e}_2$$

H) Calculate the amplitude of the magnetic field, and remember to specify units

$$B_0 = E_0/c = \frac{30Vm^{-1}}{3 \times 10^8 ms^{-1}} = 10^{-7} Vsm^{-2}$$

I) Give an expression for, and calculate the irradiance

$$I = \frac{1}{2}\epsilon_0 n c E_0^2 = \frac{1}{2}\ 8.854 \times 10^{-12}(As/Vm)\ 3 \times 10^8 m/s\ 900V^2/m^2 \approx 1.2W/m^2$$

Problem 2:
Consider the plane wave traveling in free space with the wave vector $\vec{k} = 2\pi(1,1,0)/(\sqrt{2}\lambda)$, where $\lambda = 800\text{nm}$, and with an electric field amplitude of $E_0 = 3 \times 10^8\text{V/m}$. The phase of the wave is $\pi/2$ at the origin (i.e. the magnitude of the electric field vector at $x = 0, t = 0$ is zero). Direction of the electric field oscillation is given by a unit vector $\hat{e}$.

A) Find an expression for the wave using a real representation. State the condition for $\hat{e}$ that must be satisfied to ensure that divergence of the electric field vanishes.

$$\hat{e}E_0 \sin[\vec{k}.\vec{r} - \omega t] \quad \text{where} \quad \omega = \frac{2\pi c}{\lambda} \quad \text{and} \quad \hat{e}.\vec{k} = 0$$

B) Find an expression for the wave using the complex notation.

$$\hat{e}E_0 \exp[i(\vec{k}.\vec{r} - \omega t - \pi/2)]$$

C) Find the frequency ν of the wave in THz.

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8 m s^{-1}}{8 \times 10^{-7} m} = 0.375 \times 10^{15} s^{-1}$$

D) Find a point other than the origin where the phase is 2π at $t = 0$

$$\text{any point for which} \quad \vec{k}.\vec{r} = 2\pi \quad \text{for example} \quad \vec{r} = (\sqrt{2}\lambda, 0, 0)$$

E) Describe (either using words or by giving an equation) the phase fronts of this wave

1. phase front = any plane that is perpendicular to $\vec{k}$
2. any plane parallel to the plane given by $x + y = 0$
3. $\vec{k}.\vec{r} = \text{const}$

F) Give two possible choices for $\hat{e}$ (so that divergence of the field is zero)

$$\hat{e}_1 = (1, -1, 0)/\sqrt{2} \qquad \hat{e}_2 = (0, 0, 1)$$

G) For your second choice in F), specify the unit vector $\hat{b}$ for the direction of oscillation of the magnetic field

$$\hat{b} = (-1, -1, 0)/\sqrt{2} = +\hat{e}_1$$

H) Calculate the amplitude of the magnetic field, and remember to specify units

$$B_0 = E_0/c = \frac{3 \times 10^8 V m^{-1}}{3 \times 10^8 m s^{-1}} = 1 V s m^{-2}$$

I) Give an expression for, and calculate the irradiance

$$I = \frac{1}{2}\epsilon_0 n c E_0^2 = \frac{1}{2}\, 8.854 \times 10^{-12} (As/Vm)\; 3 \times 10^8 m/s\; 9 \times 10^{16} V^2/m^2 \approx 1.2 \times 10^{14} W/m^2$$

Problem 3:
Which of these pairs **can** represent an electromagnatic plane wave propagating in free space? If not, say why. If yes, specify sufficient condition(s) that must be satisfied. Note that such conditions need not be unique — it is OK to choose an easy way out.

A)
$$\vec{E} = \hat{\imath} E_0 \cos[kx - \omega t] \qquad \vec{B} = \hat{\jmath} E_0/c \cos[kx - \omega t]$$

NO: not transverse

B)
$$\vec{E} = \hat{k} E_0 \cos[kx - \omega t] \qquad \vec{B} = \hat{\jmath} E_0/c \cos[kx - \omega t]$$

NO: wrong sign of magnetic field

C)
$$\vec{E} = \hat{\imath} E_0 \exp[i(kz - \omega t)] \qquad \vec{B} = \hat{\jmath} B_0 \exp[i(kz - \omega t)]$$

YES: If $B_0 = E_0/c$ and $k = \omega/c$

C)
$$\vec{E} = \hat{e} E_0 \exp[i(u(z + y + x) - \omega t)] \qquad \vec{B} = \hat{b} B_0 \exp[i(u(x + y + z) - \omega t)]$$

YES: If $B_0 = E_0/c$, $\hat{e} = (1, -1, 0)/\sqrt{2}$, $\hat{b} = (-1, -1, +2)/\sqrt{6}$, and $\sqrt{3}u = \omega/c$

D)
$$\vec{E} = \hat{\jmath} E_0 \cos[kx - \omega t] \qquad \vec{B} = \hat{k} E_0/c \sin[kx - \omega t]$$

NO: wrong combination of cos and sin, fields do not oscillate "together"

E)
$$\vec{E} = \hat{e} E_0 \exp[i\frac{2\pi}{\lambda}(x - y - ct)] \qquad \vec{B} = \hat{b} B_0 \exp[i\frac{2\pi}{\lambda}(x - y - ct)]$$

NO : wrong propagation speed

E)
$$\vec{E} = \hat{k} E_0 \exp[i(\frac{2\pi}{\sqrt{2}\lambda}(x - y - \sqrt{2}ct) + \pi)] \qquad \vec{B} = (\hat{\imath} + \hat{\jmath}) E_0/(\sqrt{2}c) \exp[i\frac{2\pi}{\sqrt{2}\lambda}(x - y - \sqrt{2}ct)]$$

YES: as is

Problem 4:
Given are electric and magnetic fields:

$$\vec{E} = \hat{k}E_0 \exp[i\frac{2\pi}{\sqrt{2}\lambda}(x+y-\sqrt{2}ct)] \qquad \vec{B} = (\hat{i}-\hat{j})E_0/(\sqrt{2}c)\exp[i\frac{2\pi}{\sqrt{2}\lambda}(x+y-\sqrt{2}ct)]$$

A) State the Ampere's law (in differential form) for free space

$$\nabla \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E}$$

B) Demonstrate by direct calculations that the above fields do satisfy the equation you gave in A)

First, it will be convenient to rewrite the given fields as follows:

$$\vec{E} = \hat{k}E_0 \exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)] \qquad \vec{B} = \frac{(\hat{i}-\hat{j})}{\sqrt{2}}(E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]$$

where we identify the wave vector and a unit vector pointing in the same direction:

$$\vec{k} = \frac{2\pi}{\lambda}\hat{n} \qquad \hat{n} = (1,1,0)/\sqrt{2}\ .$$

Using operator equivalencies, calculate LHS:

$$\nabla \times \vec{B} = \nabla \times \frac{(\hat{i}-\hat{j})}{\sqrt{2}}(E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)] = i\vec{k} \times \frac{(\hat{i}-\hat{j})}{\sqrt{2}}(E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]$$

Here it is fastest to express $\vec{k}$ with the help of $\hat{n}$ and use algebra of $\hat{i}, \hat{j}, \hat{k}$ to evaluate the cross product (do not bother with a determinant or similar)

$$\nabla \times \vec{B} = i\frac{2\pi}{\lambda}\frac{1}{2}\left[(\hat{i}+\hat{j}) \times (\hat{i}-\hat{j})\right](E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)] = -i\hat{k}\frac{2\pi}{\lambda}(E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]\ .$$

Now evaluate RHS:

$$\frac{1}{c^2}\partial_t \vec{E} = \frac{1}{c^2}\frac{-i2\pi c}{\lambda}\hat{k}E_0 \exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)] = -i\frac{2\pi}{\lambda}\hat{k}(E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]$$

So we have

$$LHS = RHS$$

Problem 5:
Given are electric and magnetic fields:

$$\vec{E} = \hat{k} E_0 \exp[i\frac{2\pi}{\sqrt{2}\lambda}(x + y - \sqrt{2}ct)] \qquad \vec{B} = (\hat{\imath} - \hat{\jmath})E_0/(\sqrt{2}c)\exp[i\frac{2\pi}{\sqrt{2}\lambda}(x + y - \sqrt{2}ct)]$$

A) State the Faraday's law for free space

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

B) Demonstrate by direct calculations that the above fields do satisfy the equation you gave in A)

First, it will be convenient to rewrite the given fields as follows:

$$\vec{E} = \hat{k} E_0 \exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)] \qquad \vec{B} = \frac{(\hat{\imath} - \hat{\jmath})}{\sqrt{2}}(E_0/c)\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]$$

where

$$\vec{k} = \frac{2\pi}{\lambda}\hat{n} \qquad \hat{n} = (1,1,0)/\sqrt{2}\ .$$

Using operator equivalencies, calculate LHS:

$$\nabla \times \vec{E} = i\frac{2\pi}{\lambda}\frac{1}{\sqrt{2}}\left[(\hat{\imath} + \hat{\jmath}) \times \hat{k}\right] E_0 \exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)] = i\frac{2\pi}{\lambda}\frac{\hat{\imath} - \hat{\jmath}}{\sqrt{2}} E_0 \exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]\ .$$

Now evaluate RHS:

$$-\partial_t \vec{B} = i\frac{2\pi}{\lambda}(\hat{\imath} - \hat{\jmath})E_0/(\sqrt{2})\exp[i(\vec{k}.\vec{r} - \frac{2\pi c}{\lambda}t)]\ .$$

So we have

$$LHS = RHS$$