Problem 1:
The electric and magnetic fields representing a plane wave (propagating in vacuum) are specified as follows:

$$\vec{E} = \hat{\imath} E_0 \cos[kz - \omega t] \qquad \vec{B} = \hat{\jmath} B_0 \cos[kz - \omega t]$$

with $E_0 = 10^9 V/m$ and $\omega = 5 \times 10^{15} s^{-1}$.
A) Identify the wave vector $\vec{k}$ of this plane wave and express it, using the dispersion relation, in terms of ω

$$\vec{k} = \hat{k}\frac{\omega}{c}$$

B) Use complex representation plus operator uquivalencies, and Ampere's law to show that $B_0 = E_0/c$

$$\nabla \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E}$$

$$i\frac{\omega}{c}\hat{k} \times \hat{\jmath} B_0 \exp[i(kz - \omega t)] = \frac{1}{c^2}(-i\omega)\hat{\imath} E_0 \exp[i(kz - \omega t)]$$

$$\frac{\omega}{c}(-\hat{\imath})B_0 = \frac{1}{c^2}\omega(-\hat{\imath})E_0$$

$$B_0 = \frac{E_0}{c}$$

C) Calculate the instantaneous Poynting vector $\vec{S}$ at the origin of coordinate system, $x, y, z = 0$ for a general time t

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{\mu_0}\hat{\imath} E_0 \cos[kz - \omega t] \times \hat{\jmath} B_0 \cos[kz - \omega t] = \frac{1}{\mu_0} E_0 B_0 (\hat{\imath} \times \hat{\jmath}) \cos^2[kz - \omega t]$$

$$\vec{S}(x = 0, y = 0, z = 0) = \hat{k}\frac{\epsilon_0}{\mu_0 \epsilon_0}\frac{E_0^2}{c}\cos^2[\omega t] = \hat{k} c \epsilon_0 E_0^2 \cos^2[\omega t]$$

D) Calculate the time-averaged Poynting vector $\langle \vec{S} \rangle$ and

Using the fact that the mean value of $\cos^2$ is equal to one half, we get

$$\langle \vec{S} \rangle = \hat{k}\frac{1}{2} c \epsilon_0 E_0^2 = \hat{k}\frac{1}{2} 3 \times 10^8 m/s \; 8.854 \times 10^{-12} (As/Vm) \; 10^{18} (V/m)^2 \approx \hat{k} 1.3 \times 10^{15} AV/m^2$$

E) What energy corresponds to a single photon in this plane wave?

$$E_{ph} = \hbar\omega = 1 \times 10^{-34} Js \; 5 \times 10^{15} s^{-1} = 5 \times 10^{-19} J$$

E) Calculate the photon flux Φ (i.e. the number of photons crossing a given area per unit of time) over an area of $A = 100 \mu m^2$

Using

$$\Phi E_{ph} = A \langle S \rangle$$

we get

$$\Phi = \frac{c}{2\hbar\omega}\epsilon_0 E_0^2 A = \frac{1.3 \times 10^{15} AV/m^2}{5 \times 10^{-19} J} 100 \times 10^{-12} m^2 \approx 2.6 \times 10^{23} s^{-1}$$

Problem 2:
The electric and magnetic fields representing a plane wave (propagating in vacuum) are specified as follows:

$$\vec{E} = \hat{k} E_0 \sin[\frac{2\pi}{\lambda}\hat{n}.\vec{r} - \frac{2\pi c}{\lambda}t] \qquad \vec{B} = \frac{\hat{i}-\hat{j}}{\sqrt{2}} B_0 \sin[\frac{2\pi}{\lambda}\hat{n}.\vec{r} - \frac{2\pi c}{\lambda}t]$$

with $E_0 = 1V/m$ and $\lambda = 1\mu m$ and $\hat{n} = (1,1,0)/\sqrt{2}$
A) Explain the meaning of $\hat{n}$

$\hat{n}$: unit vector, direction of propagation

B) Use complex representation and Faraday's law to show that $cB_0 = E_0$

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

$$i\frac{2\pi}{\lambda}\frac{1}{\sqrt{2}}(\hat{i}+\hat{j}) \times \hat{k} E_0 \exp[i(\frac{2\pi}{\lambda}\hat{n}.\vec{r} - \frac{2\pi c}{\lambda}t - \pi/2)] = +i\frac{2\pi c}{\lambda}\frac{(\hat{i}-\hat{j})}{\sqrt{2}} B_0 \exp[i(\frac{2\pi}{\lambda}\hat{n}.\vec{r} - \frac{2\pi c}{\lambda}t - \pi/2)]$$

... expand the cross product and cancel to get

$$E_0 = cB_0$$

C) Calculate the instantaneous Poynting vector $\vec{S}$ at the origin of coordinate system, $x, y, z = 0$ for a general time t

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{\mu_0}\hat{k} \times \frac{\hat{i}-\hat{j}}{\sqrt{2}} E_0 B_0 \sin^2[\ldots]$$

$$\vec{S}(x=0, y=0, z=0) = \frac{\hat{i}+\hat{j}}{\sqrt{2}}\frac{\epsilon_0}{\mu_0\epsilon_0}\frac{E_0^2}{c}\sin^2[\frac{2\pi c}{\lambda}t] = \hat{n} c\epsilon_0 E_0^2 \sin^2[\omega t]$$

D) Calculate the time-averaged Poynting vector $\langle\vec{S}\rangle$ and

Using the fact that the mean value of $\sin^2$ is equal to one half, we get

$$\langle\vec{S}\rangle = \hat{n}\frac{1}{2}c\epsilon_0 E_0^2 = \frac{1}{2} 3 \times 10^8 m/s \; 8.854 \times 10^{-12}(As/Vm) \; 1(V/m)^2 = 1.3 \times 10^{-5} W/m^2$$

E) What energy corresponds to a single photon in this plane wave?

$$E_{ph} = \hbar\omega = \hbar\frac{2\pi c}{\lambda} = 1 \times 10^{-34} Js \; \frac{2\pi \; 3 \times 10^8 m/s}{10^{-6} m} \approx 1.9 \times 10^{-19} J$$

E) Calculate the photon flux Φ (i.e. the number of photons crossing an area per unit of time) over an area of $A = 1m^2$
Using

$$\Phi E_{ph} = A\langle S\rangle$$

we get

$$\Phi = \frac{c}{2\hbar\omega}\epsilon_0 E_0^2 A = \frac{13.3 \times 10^{-4} AV/m^2}{1.9 \times 10^{-19} J} 1m^2 \approx 7 \times 10^{15} s^{-1}$$