Example: Extraordinary wave index, dependence on direction of propagation

This problem example shows calculation of the refractive index experienced by a plane wave propagating in a general direction through a uni-axial crystal.

Consider a uni-axial crystal with ordinary and extraordinary refractive indices n_o and n_e. A plane wave is propagating at an angle α with respect to the optic axis. Assume that the optic axis points along the z-axis of the coordinate system, and that the propagation vector lies in the $y - z$ plane.

Your task is to demonstrate that the refractive index depends on α as

$$n(\alpha) = \frac{n_o n_e}{\sqrt{n_e^2 \cos^2\alpha + n_o^2 \sin^2\alpha}}$$

Hint:

You will need to use all Maxwell equations, plus the medium constitutive relation,

$$D_x = \epsilon_0 \epsilon_x E_x \qquad D_y = \epsilon_0 \epsilon_y E_y \qquad D_z = \epsilon_0 \epsilon_z E_z$$

$$\epsilon_x = \epsilon_y = n_o^2 \qquad \epsilon_z = n_e^2$$

Lesson: The following steps show that the refractive index is obtained from the Maxwell as a solvability condition in the form of the angle-dependent dispersion relation:

$$k^2 = \frac{\omega^2 n(\alpha)^2}{c^2}$$

A) What is the unit vector $\hat{n}$ in the direction of propagation?

Answer:

$$\hat{n} = (0, \sin\alpha, \cos\alpha)$$

B) Assuming that the wave is ordinary, determine the unit polarization direction vector $\hat{o}$

Answer:

$$\hat{o} = (1, 0, 0);$$

C) In the rest of the problem we assume that the wave is polarized extraordinary. Determine the polarization direction vector $\hat{d}$ along which $\vec{D}$ oscillates.

Answer:
We know that $\nabla.\vec{D} = 0$ which implies that $\hat{d}.\hat{n} = 0$. Moreover, extraordinary wave has polarization perpendicular to that of ordinary, which means that $d_x = 0$. These two properties imply (up to a sign):

$$\hat{d} = (0, -\cos\alpha, +\sin\alpha)$$

D) Let the amplitude of $\vec{D}$ is D_0. Denote ω and k the angular frequency and the magnitude of the propagation vector. Write down the expression for the non-zero components of $\vec{D}$.

Answer:

$$D_y = -D_0 \cos\alpha \exp[-i\omega t + k\hat{n}.\vec{r}] \qquad D_z = +D_0 \sin\alpha \exp[-i\omega t + k\hat{n}.\vec{r}]$$

E) Write down non-zero components of the electric field

Answer:

$$E_y = -D_0/(\epsilon_0 n_o^2) \cos\alpha \exp[-i\omega t + k\hat{n}.\vec{r}] \qquad E_z = +D_0/(\epsilon_0 n_e^2) \sin\alpha \exp[-i\omega t + k\hat{n}.\vec{r}]$$

F) Using Faraday, calculate the nonzero components of $\vec{B}$

Answer:

$$-\partial_t B_x = \partial_y E_z - \partial_z E_y$$

$$i\omega B_x = ik\hat{n}_y E_z - ik\hat{n}_z E_y = ik \sin\alpha E_z - ik\cos\alpha E_y$$

$$B_x = \frac{kD_0}{\epsilon_0 \omega}\left[\frac{\cos^2\alpha}{n_o^2} + \frac{\sin^2\alpha}{n_e^2}\right] \exp[-i\omega t + k\hat{n}.\vec{r}]$$

G) Next, insert previous results in the Ampere. You should obtain two equivalent equations. The y component, for example, gives the following:

Answer:

$$\partial_t D_y = \frac{1}{\mu_0}\partial_z B_x$$

$$-i\omega D_y = \frac{1}{\mu_0} ik\hat{n}_z B_x$$

$$+D_0 \cos\alpha \exp[-i\omega t + k\hat{n}.\vec{r}] = \cos\alpha \frac{k^2 D_0}{\epsilon_0 \mu_0 \omega^2}\left[\frac{\cos^2\alpha}{n_o^2} + \frac{\sin^2\alpha}{n_e^2}\right]\exp[-i\omega t + k\hat{n}.\vec{r}]$$

$$1 = \frac{k^2 c^2}{\omega^2}\left[\frac{\cos^2\alpha}{n_o^2} + \frac{\sin^2\alpha}{n_e^2}\right]$$

H) Finally, parametrizing propagation number k in terms of the (angle-dependent) refractive index $n(\alpha)$ as

$$k^2 = n(\alpha)^2\omega^2/c^2 \ ,$$

obtain the dependence on the angle of propagation:

$$n(\alpha) = \frac{n_o n_e}{\sqrt{n_e^2\cos^2\alpha + n_o^2\sin^2\alpha}}$$