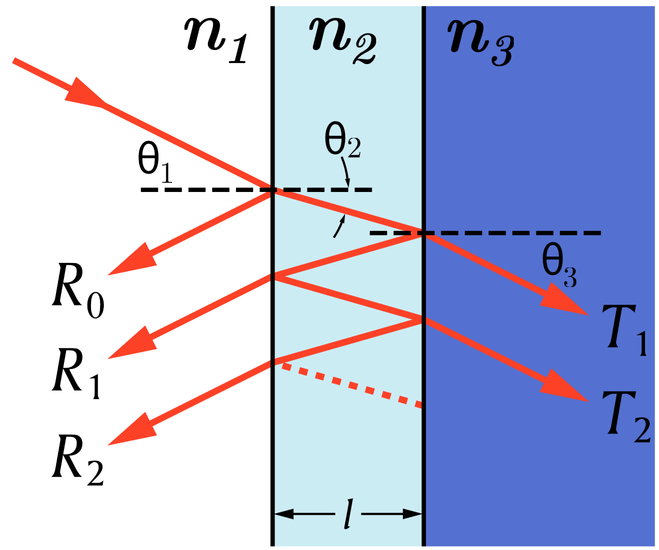
Coatings and Multi-layers

Fabry-Perot geometry is crucial for many devices. Here we look at (anti-reflective) coatings, which are essentially miniature FPs.

Plan:

- Apply what we learned about FP. We will need a bit of a generalization...



- Develop an intuitive picture of "what happens inside" of a multilayer. This will rely on the behaviors described by Fresnel equations, and in particular on phase changes experienced at reflections.

Three-layer structure: Transfer matrix treatment

For simplicity and concreteness, consider the TE polarized case.
This choice does not matter too much:
1. TM and TE results are identical for normal incidence
2. it often sufficient to work with near-normal angles

The transfer matrix representing the three-layer structure is

$$M = M^{(23)} M^{(slab)} M^{(12)}$$

$$M^{(12)} = \frac{1}{2}\begin{pmatrix} 1+a_{12} & 1-a_{12} \\ 1-a_{12} & 1+a_{12} \end{pmatrix} \qquad M^{(slab)} = \begin{pmatrix} e^{+ik_z^{(2)}l} & 0 \\ 0 & e^{-ik_z^{(2)}l} \end{pmatrix} \qquad M^{(23)} = \frac{1}{2}\begin{pmatrix} 1+a_{23} & 1-a_{23} \\ 1-a_{23} & 1+a_{23} \end{pmatrix}$$

$$a_{12} = \frac{k_z^{(1)}}{k_z^{(2)}} = \frac{n_1 \cos\theta_1}{n_2 \cos\theta_2} \approx \frac{n_1}{n_2} \qquad a_{23} = \frac{k_z^{(2)}}{k_z^{(3)}} = \frac{n_2 \cos\theta_2}{n_3 \cos\theta_3} \approx \frac{n_1}{n_2}$$

where $\approx$ denotes approximate relations usable for nearly-normal incidence

. . . matrix multiplication gives

$$M = M^{(23)} M^{(slab)} M^{(12)} = \frac{1}{2} \begin{pmatrix} 1 + a_{23} & 1 - a_{23} \\ 1 - a_{23} & 1 + a_{23} \end{pmatrix} \begin{pmatrix} e^{+ik_z^{(2)} l} & 0 \\ 0 & e^{-ik_z^{(2)} l} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 + a_{12} & 1 - a_{12} \\ 1 - a_{12} & 1 + a_{12} \end{pmatrix}$$

$$M = \frac{1}{2} \begin{pmatrix} (1 + a_{12}a_{23}) \cos(k_z^{(2)} l) + i(a_{12} + a_{23}) \sin(k_z^{(2)} l) & (1 - a_{12}a_{23}) \cos(k_z^{(2)} l) - i(a_{12} - a_{23}) \sin(k_z^{(2)} l) \\ (1 - a_{12}a_{23}) \cos(k_z^{(2)} l) + i(a_{12} - a_{23}) \sin(k_z^{(2)} l) & (1 + a_{12}a_{23}) \cos(k_z^{(2)} l) - i(a_{12} + a_{23}) \sin(k_z^{(2)} l) \end{pmatrix}$$

The relation between incident, reflected, and transmitted amplitudes, expressed with this matrix is

$$\begin{pmatrix} E_t \\ 0 \end{pmatrix} = M \begin{pmatrix} E_i \\ E_r \end{pmatrix}$$

The “lower-row” component of this equation reads

$$0 = M_{21} E_i + M_{22} E_r$$

from where we obtain the amplitude reflection coefficient of the tri-layer as:

$$r_s = \frac{E_r}{E_i} = -\frac{M_{21}}{M_{22}}$$

$$r_s = -\frac{(1 - a_{12}a_{23}) \cos(k_z^{(2)} l) + i(a_{12} - a_{23}) \sin(k_z^{(2)} l)}{(1 + a_{12}a_{23}) \cos(k_z^{(2)} l) - i(a_{12} + a_{23}) \sin(k_z^{(2)} l}$$

Special cases: AR-coating

Say we wish to minimize reflection. Assuming we can choose freely two parameters, namely thickness l and the refractive index of the coating layer, is it possible to achieve zero reflectivity?
r_s suggests two options:
Either

$$\cos(k_z^{(2)} l) = 0 \quad \text{and} \quad (a_{12} - a_{23}) = 0$$

or

$$\sin(k_z^{(2)} l) = 0 \quad \text{and} \quad (1 - a_{12} a_{23}) = 0$$

The second is not useful, because $a_{12}a_{23} = a_{13}$ which does not depend on the index of refraction of the second layer (our free parameter).

Fortunately, the first is usable, and requires that

- $k_z^{(2)} l = \frac{\pi}{2}$ which means that the thickness corresponds to one quarter of wavelength
- $n_1 n_2 = n_2^2$ which means that the refractive index n_2 should be the geometric mean of the index of substrate and of the incident medium.
- if the first medium is air, $n_2 = \sqrt{n_{substrate}}$

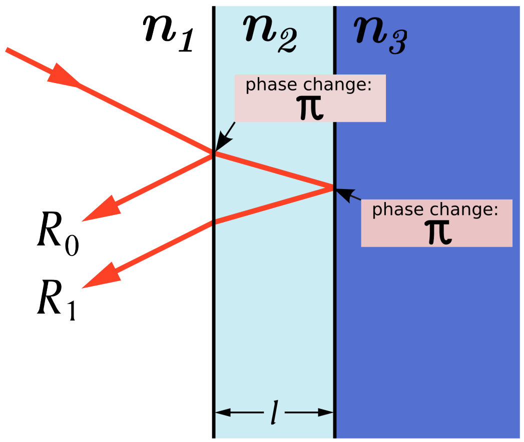
Note: Because of the frequency dependence of the refractive index, we can only achieve zero reflectivity for a single wavelength. Such a choice may not be optimal at other wavelengths...

Intuitive picture of AR-coating

Fresnel says: reflection from an optically denser medium causes a phase change of π (equivalently, change of sign).
Thus if

$$n_1 < n_2 < n_3$$

then:



If, further, we adjust the thickness of the coating layer such that

$$k_z^{(2)} l = \frac{\pi}{2} \ ,$$

the total phase difference between R_1 and R_0 will be

$$\pi + \pi/2 + \pi/2 - \pi \ ,$$

and the two waves will interfere destructively. Provided these two waves dominate the reflected field, we can see that the coating acts as a Fabry-Perot at resonance, and decreases the reflectivity of this coated surface.

Intuitive picture of AR-coating continued...

Q: Having justified, based on qualitative argument, that the thickness should be a quarter of wavelength, is it also possible to "guess" what is the optimal value of n_2?

A: The idea is to maximize potential for interference between R_0 and R_1. We have seen previously that the biggest interference effects occur for comparable field amplitudes. If R_0 and R_1 were different amplitudes, they could not destruct each other completely...

Approximation:

- each transmission is nearly perfect, with $t \approx 1$
- only R_0 and R_2 contribute to the reflected field
- incidence is normal

Under these assumptions we have

$$R_0 \approx E_0 r_{12} = E_0 \frac{n_1 - n_2}{n_1 + n_2} \qquad R_1 \approx E_0 r_{23} = E_0 \frac{n_2 - n_3}{n_2 + n_3}$$

Asking that the two are equal, leads to

$$0 = \frac{n_1 - n_2}{n_1 + n_2} - \frac{n_2 - n_3}{n_2 + n_3} = -2 \frac{n_2^2 - n_1 n_3}{(n_1 + n_2)(n_2 + n_3)} \ ,$$

... and we see that the two amplitudes are indeed equal if

$$n_2 = \sqrt{n_1 n_3}$$

Thus, both important properties of AR-coating can be argued without much calculations.

Problem example: From the transfer matrix calculation, we have obtained one of the conditions for the tri-layer reflectivity being zero as

$$\cos(k_z^{(2)} l) = 0$$

and said that it implied quarter wavelength thickness. What about other solutions with a thicker coating? Why not to use 3/4-wavelength or larger thickness?

A) Take the reflection coefficient r_s as derived, and argue that chromatic dispersion implies that r_s becomes non-zero in the immediate vicinity of the optimal wavelength as long as $\partial n_2/\partial\lambda$ is nonzero (as it always is).

B) Give an argument showing that this reflectivity detriment is lesser if

$$\frac{\partial r_s(n_2)}{\partial n_2}$$

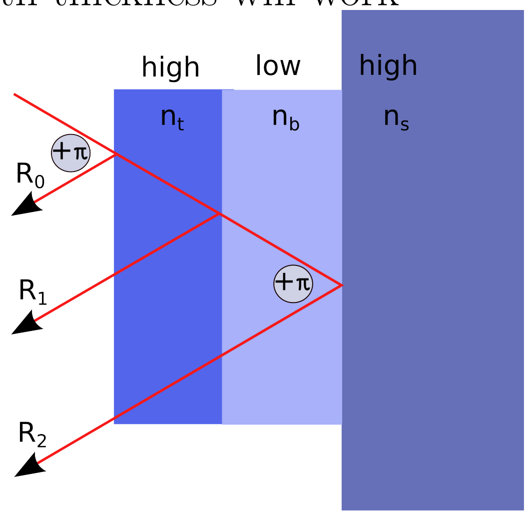
as small as possible.

C) Show, through explicit calculation, that the above becomes larger and larger for the "thicker" solutions.

High-Reflectance multi-layers

- For a high-reflectivity structure, we need at least two layers deposited on a substrate.
- From the AR discussion we know that the second layer must be lower index
- We guess that quarter-wavelength thickness will work

Intuitive picture:



The phase changes in the reflected amplitudes are:

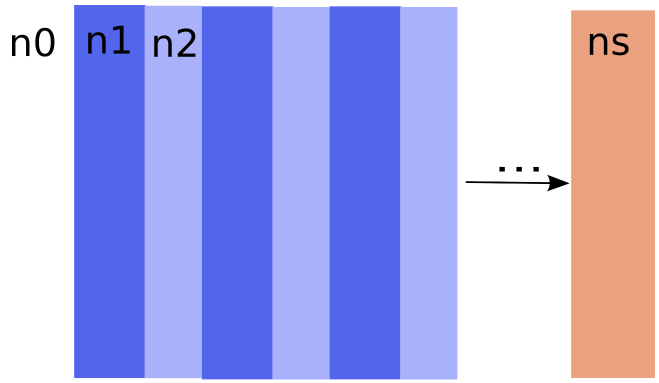
R_0: $+\pi$ due to reflection on the first, air-n_t interface

R_1: $+\pi/2 + \pi/2$ due to traveling twice through quarter-wavelength slab. No reflection phase.

R_2: $+\pi$ due to reflection off the bottom-substrate interface. 2π due to propagation through four quarter-wavelength thickness of top and bottom coating layer.

Result: Constructive interference = high reflectivity

Distributed Bragg Mirrors



Transfer matrix for N period-repeats:

$$M = M^{1s}\left(M^{(21)}M^{(2)}M^{(12)}M^{(1)}\right)^N M^{(01)}$$

Assume that layers are quarter-wavelength:

$$M^{(1)} = M^{(2)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

then

$$A = M^{(21)}M^{(2)}M^{(12)}M^{(1)} = -\frac{1}{2}\begin{pmatrix} a_{12}+a_{21} & a_{21}-a_{12} \\ a_{21}-a_{12} & a_{12}+a_{21} \end{pmatrix} = \frac{-1}{2}a_{21}\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix} + \frac{-1}{2}a_{12}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$$

We need to find A^N for

$$A = -\frac{1}{2}\begin{pmatrix} a_{12}+a_{21} & a_{21}-a_{12} \\ a_{21}-a_{12} & a_{12}+a_{21} \end{pmatrix} = -\frac{1}{2}a_{21}\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix} - \frac{1}{2}a_{12}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \equiv (-a_{21})P_e + (-a_{12})P_o$$

This is easy when you realize that the two matrices P_o, P_e on the right have pleasant properties: They are what is called orthogonal projectors:

$$P_e P_o = 0 \qquad P_e^k = P_e \qquad P_o^k = P_o$$

so it is straightforward to calculate their arbitrary (matrix) functions.
This is why we get

$$A^N = (-a_{21})^N P_e + (-a_{12})^N P_o$$

Note: Those familiar with eigenvector decomposition of matrices will recognize that $P_{e,o}$ are nothing but projectors onto even and odd eigenvector of A.

So we have the transfer matrix in the form:

$$M = M^{(1s)} A^N M^{(01)}$$

with

$$M^{(01)} = \frac{1}{2}\begin{pmatrix} 1 + a_{01} & 1 - a_{01} \\ 1 - a_{01} & 1 + a_{01} \end{pmatrix} \equiv P_e + a_{01} P_o$$

and

$$M^{(1s)} = \frac{1}{2}\begin{pmatrix} 1 + a_{1s} & 1 - a_{1s} \\ 1 - a_{1s} & 1 + a_{1s} \end{pmatrix} \equiv P_e + a_{1s} P_o$$

so that

$$M = (P_e + a_{1s} P_o)\left[(-a_{21})^N P_e + (-a_{12})^N P_o\right](P_e + a_{01} P_o)$$

$$M = P_e(1 \times (-a_{21})^N \times 1) + P_o(a_{1s} \times (-a_{12})^N \times a_{01})$$

Explicitly:

$$M = \frac{1}{2}\begin{pmatrix} (-a_{21})^N + a_{0s}(-a_{12})^N & (-a_{21})^N - a_{0s}(-a_{12})^N \\ (-a_{21})^N - a_{0s}(-a_{12})^N & (-a_{21})^N + a_{0s}(-a_{12})^N \end{pmatrix}$$

where we have used

$$a_{1s} a_{01} = a_{0s}$$

Having found the transfer matrix, we can obtain the reflection coefficient the same way as on page 3:

$$r_s = -\frac{M_{21}}{M_{22}} = -\frac{(-a_{21})^N - a_{0s}(-a_{12})^N}{(-a_{21})^N + a_{0s}(-a_{12})^N}$$

For nearly normal incidence a_{ik} reduces to ratio of refractive indices and we finally get:

$$r = -\frac{(-n_2/n_1)^N - n_0/n_s(-n_1/n_2)^N}{(-n_2/n_1)^N + n_0/n_s(-n_1/n_2)^N} = -\frac{(-n_2/n_1)^{2N} - n_0/n_s}{(-n_2/n_1)^{2N} + n_0/n_s}$$

and the reflectance is

$$R = |r|^2 = \left[\frac{(n_2/n_1)^{2N} - n_0/n_s}{(n_2/n_1)^{2N} + n_0/n_s}\right]^2$$

Note:

- as $n_2 > n_1$ the reflectance tends to unity for a large number of periods
- in that limit, influence of the incident n_0 and substrate refractive index n_s is insignificant
- this expression can only be applied at the "design wavelength" for which the layers are quarter-wavelength (which we have assumed)
- the result applies to TE and TM polarizations (because nearly-normal incidence was also assumed)

Exercise:

Perform sanity check of the result for the multilayer reflectivity. Assume that the refractive indices of the alternating layers are equal. The situation is then the same as for a three-layer structure. Verify that the result agrees with the prediction for three-layer r_s on page 3.

Hint: pay attention to the total thickness of the layer...