Faraday rotation: Oscillator model

The following is a simplified model, in which we neglect absorption, and also treat the magnetic field as a small perturbation. (This treatment can, of course, be improved — see book for a slightly more accurate result.)

Start from the equation of motion for an oscillator exposed to:
1. Circularly polarized EM wave
2. Constant magnetic field (oriented along the propagation direction)

$$m\frac{d^2\vec{r}(t)}{dt^2} + k\vec{r} = -e\vec{E}(t) - e\frac{d\vec{r}(t)}{dt} \times \vec{B}$$

choose: $\vec{B} = (0, 0, B)$ and $\vec{E} = (E_x, E_y, 0)e^{-i\omega t}$, and expect the solution in the form $\vec{r} = (r_x, r_y, 0)e^{-i\omega t}$.

$$-m\omega^2\vec{r} + k\vec{r} = -e\vec{E} + ie\omega\vec{r} \times \vec{B}$$

What we need is actually the polarization, i.e. $\vec{P} = -e\vec{r}N$, so let us write the corresponding equation:

$$-m\omega^2\vec{P} + k\vec{P} = e^2N\vec{E} + ie\omega\vec{P} \times \vec{B}$$

the oscillator eigen-frequency is $\omega_0^2 = k/m$, so we have

$$(\omega_0^2 - \omega^2)\vec{P} = \frac{e^2N}{m}\vec{E} + \frac{ie}{m}\omega\vec{P} \times \vec{B}$$

Next, switch to the component form:

$$(\omega_0^2 - \omega^2)\vec{P} = \frac{e^2 N}{m}\vec{E} + \frac{ie}{m}\omega\vec{P} \times \vec{B} = \frac{e^2 N}{m}\vec{E} + \frac{ieB}{m}\omega(P_x\hat{x} + P_y\hat{y}) \times \hat{z}$$

$$(\omega_0^2 - \omega^2)P_x = \frac{e^2 N}{m}\vec{E}_x + \frac{ieB}{m}\omega P_y \qquad (\omega_0^2 - \omega^2)P_y = \frac{e^2 N}{m}\vec{E}_y - \frac{ieB}{m}\omega P_x$$

Approximation: Assume B is weak, so that it only slightly perturbs the motion of the electron cloud. Solve first for $B = 0$:

$$P_x^{(0)} = \frac{Ne^2}{m}\frac{E_x}{\omega_0^2 - \omega^2} \qquad P_y^{(0)} = \frac{Ne^2}{m}\frac{E_y}{\omega_0^2 - \omega^2}$$

This is the un-perturbed solution. To obtain a correction due to the magnetic field, we insert it in the terms that contain B, e.g. for the x-component we get:

$$(\omega_0^2 - \omega^2)P_x = \frac{e^2 N}{m}E_x + \frac{ieB}{m}\omega P_y^{(0)} = \frac{e^2 N}{m}E_x + \frac{ieB}{m}\frac{Ne^2}{m}\frac{\omega}{\omega_0^2 - \omega^2}E_y$$

$$P_x = \frac{e^2 N}{m}\frac{1}{(\omega_0^2 - \omega^2)}E_x + \frac{ieB}{m}\frac{Ne^2}{m}\frac{\omega}{(\omega_0^2 - \omega^2)^2}E_y \equiv \epsilon_0\chi_{11}E_x + \epsilon_0\chi_{12}E_y$$

from where we simply read the non-diagonal element of the susceptibility tensor:

$$\chi_{12} = \frac{ieB}{m}\frac{Ne^2}{m\epsilon_0}\frac{\omega}{(\omega_0^2 - \omega^2)^2}$$

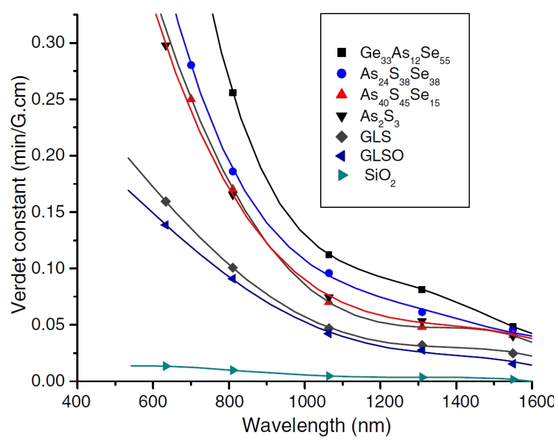
Using our previous result on the relation between Verdet constant and χ_{12} we get the specific rotatory power (angle per length of propagation):

$$\delta = \frac{\chi_{12}\pi}{n_0\lambda} = \frac{eB}{m}\frac{Ne^2}{m\epsilon_0 n_0}\frac{\pi}{\lambda}\frac{\omega}{(\omega_0^2 - \omega^2)^2}$$

Observations:

- As expected from the intuitive argument given before, χ_{12} is proportional to B
- This formula says that away from resonance, Verdet constant decreases with wavelength:

chalcogenide glasses:



water:

λ nm	V_B (Rad/Tm)
347	12.4
458	6.78
488	5.85
514	5.24
580	4.1
596	3.81
633	3.35
694	2.66

- Specific rotatory power increases with chromatic dispersion. The above can be re-written as this so-called Becquerel equation:

$$\delta = -\frac{eB}{2m}\frac{\lambda}{c}\frac{dn}{d\lambda}$$