Recap material from the last session:

## Jones Calculus: Polarization Vectors

State of polarization represented by *complex* amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ (assuming propagation along $z$).
Relation to the polarization ellipse:

$$\phi_{yx} = \arg(\mathcal{E}_x/\mathcal{E}_y) \qquad\qquad (A_x/A_y) = \left|\frac{\mathcal{E}_x}{\mathcal{E}_y}\right|$$

## Jones Vector:

$$\bar{\mathcal{E}}(z) = \begin{pmatrix} \mathcal{E}_x(z) \\ \mathcal{E}_y(z) \end{pmatrix}$$

A complex-valued, two-component column vector:

- Usually normalized to unity, but not always
- It is the **ratio of components** that is important
- Can factor-out arbitrary complex factor
- In problems dealing with transmitted power, normalization **is important**
- Power in the beam = norm$^2$ of the Jones vector:

$$< \bar{\mathcal{E}}|\bar{\mathcal{E}} >= (\bar{\mathcal{E}}^T)^*.\bar{\mathcal{E}} = \bar{\mathcal{E}}^+.\bar{\mathcal{E}} = \mathcal{E}_x^*\mathcal{E}_x + \mathcal{E}_y^*\mathcal{E}_y = |\mathcal{E}_x|^2 + |\mathcal{E}_y|^2$$

## Jones Calculus: Polarization Vectors

- $x$-polarized:

$$\bar{\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $y$-polarized:

$$\bar{\mathcal{E}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- RHC, right circular polarization:

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- LHC, left circular polarization:

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}$$

- linear polarization, +45 deg

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- linear polarization, $-45$ deg

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

## Jones Calculus: Matrices for Optical Elements

Polarizers:

- Linear polarizer, transmission horizontal ($x$)

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Linear polarizer, transmission vertical ($y$)

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Linear polarizer, transmission axis at $\pm 45$ deg

$$M = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

- Circular polarizer, Right and Left:

$$M = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad M = \frac{1}{2}\begin{pmatrix} 1 & -i \\ +i & 1 \end{pmatrix}$$

**Note:** Polarizer matrices have zero determinants. Why?

## Jones Calculus: Matrices for Optical Elements

Plates:

- Quarter wave-plate, fast axis vertical:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

- Quarter wave-plate, fast axis horizontal:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & +i \end{pmatrix}$$

- Quarter wave-plate, fast at $\pm 45$ deg

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$$

- Half wave-plate, fast axis horizontal or vertical

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**Note:** Plates have $|\det M| = 1$, i.e. they are *unitary* operators.

## Rotated optical elements (and their Jones matrices):

Consider a Jones vector

$$\bar{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

in a rotated basis system $\hat{e}'_x, \hat{e}'_y$

$$\begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix} = R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^{-1}(-\theta)$$

Take

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_{z=L} = M \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_{z=0}$$

and represent the same in the rotated frame of reference:

$$\begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix}_L = R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_L = R(\theta) M \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_0 = R(\theta) M R^{-1}(\theta) R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_0 \equiv M' R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_0 = M' \begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix}_0$$

so the Jones matrix in the rotated system is

$$M' = R(\theta) M R^{-1}(\theta)$$

## Rotated optical elements (and their Jones matrices):

Assume it is the element $M'$ in the rotated system that we know. Then back in the lab frame we have:

$$M = R^{-1}(\theta)M'R^(\theta) = R(-\theta)M'R(\theta)$$

**Example:** Rotated polarizer
In the rotated frame it is

$$M' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

What is it in the lab frame:

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

Special case: $\theta = \pm 45^o$

$$M = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

Intensity behind a polarizer (as a function of its orientation)
Consider $x$-polarized light incident on a rotated polarizer:

$$\bar{\mathcal{E}}(L) = M\bar{\mathcal{E}}(0)$$

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = E_0 \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} E_0\cos^2\theta \\ E_0\sin\theta\cos\theta \end{pmatrix} \; ,$$

so the intensity is

$$I(L) = I_x(L) + I_y(L) = I_0\cos^4(\theta) + I_0\cos^2(\theta)\sin^2(\theta) = I_0\cos^2\theta$$

For the initially **unpolarized** light, one half of the power transmits beyond the polarizer, so after the analyzer we have

$$I(L) = \frac{1}{2}I_0\cos^2\theta$$

This is called **Malus‘s law**.

**Example:** Consider a LHC polarized wave incident on a rotated polarizer:

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} \mathcal{E}_x(0) \\ \mathcal{E}_y(0) \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2\theta + i\sin\theta\cos\theta \\ \sin\theta\cos\theta + i\sin^2\theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta e^{i\theta} \\ \sin\theta e^{i\theta} \end{pmatrix} = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

This means that

$$\phi_{yx}(L) = \arg(\cot\theta) = 0, \pi$$

$$\left(\frac{A_y}{A_x}\right) = \tan\theta$$

So the action is: Circular $\rightarrow$ Linear

**Q:** Do this for "any" incident polarization state and show that the result is "the same" with one exception.

## Phase retarders and wave-plates

Consider the situation investigated in the example with the uni-axial medium, and propagation in a direction perpendicular to the optic axis. The two polarizations (parallel and perpendicular to c) each experience different refractive index. As a result, they will accumulate different phase changes:

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \begin{pmatrix} \mathcal{E}_x(0)e^{i\phi_x} \\ \mathcal{E}_y(0)e^{i\phi_y} \end{pmatrix}$$

which can be also written with this diagonal (Jones) matrix:

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix} \begin{pmatrix} \mathcal{E}_x(0) \\ \mathcal{E}_y(0) \end{pmatrix}$$

Because only the phase difference is important,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\phi_y-\phi_x)} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{pmatrix}$$

$$\phi_x = \frac{2\pi}{\lambda} n_x L \qquad \phi_y = \frac{2\pi}{\lambda} n_y L \qquad \Delta\phi = \frac{2\pi}{\lambda}(n_y - n_x)L$$

So, depending on the thickness of the medium, we have:

- $\Delta\phi = \pm\pi/2$ — quarter-wave plate
- $\Delta\phi = \pi$ — half-wave plate
- $\Delta\phi = 2\pi$ — full-wave plate

# Transformation of polarization

## Quarter-wave plate

for initial linear polarization at $45^o$ w.r.t. fast/slow axis

$$\bar{\mathcal{E}}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\bar{\mathcal{E}}(0) = \begin{pmatrix}1 & 0\\0 & \pm i\end{pmatrix}\bar{\mathcal{E}}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm i\end{pmatrix}$$

So, this is: Linear $\rightarrow$ Circular

## Half-wave plate

for the same initial polarization

$$\bar{\mathcal{E}}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\bar{\mathcal{E}}(0) = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\bar{\mathcal{E}}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

So, this is: Net rotation of Linear by ninety degs.

## Transformation of polarization
## Half-wave plate

for the initial LHC polarization

$$\bar{\mathcal{E}}(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\bar{\mathcal{E}}(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{\mathcal{E}}(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

So, this is: LHC $\rightarrow$ RHC

## Transformation of polarization

Consider two crossed polarizers, so that no light gets through. Now insert between them a third polarizer oriented at $45^o$ (thus halving the angle). What is the transmission now?

Solution: The Jones matrix of the system is

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \sin\theta\cos\theta & 0 \end{pmatrix}$$

So there is in general no-zero transmission. It will be largest if $\theta$ is $45^o$.

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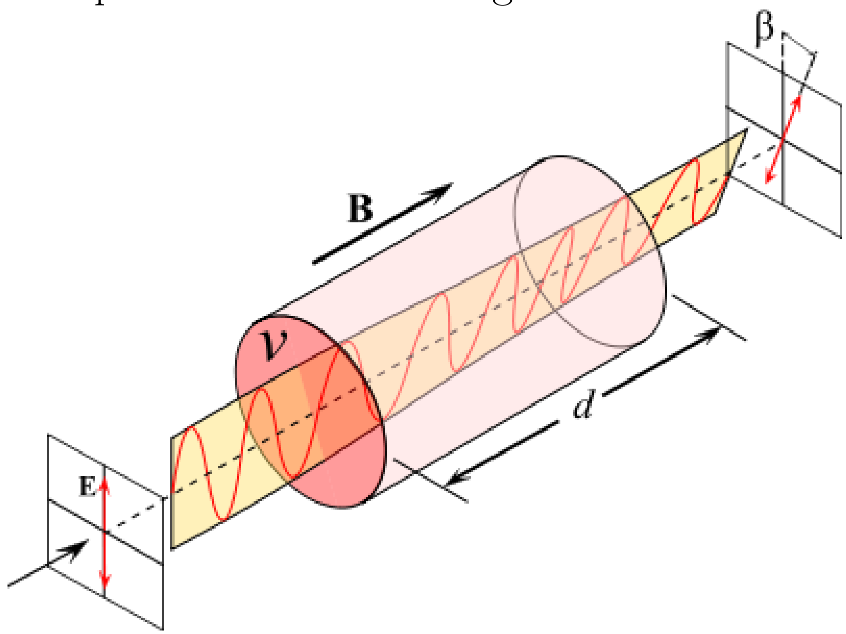
**Exercise:** Explain what happens to the transmission when we keep inserting more polarizers, always oriented such that they divide the angle between the orientations of their neighbors. What happens in the limit of infinitely many polarizers?

**Hint:** It is easiest to solve this problem with the help of a drawing...

## Faraday Effect

Strong magnetic field, oriented along the propagation direction of an EM wave breaks the symmetry between left- and right-handed spiral. This allows for gradual rotation of the polarization direction:




Rotation angle proportional to the magnetic field strength

$$\beta = VBd$$

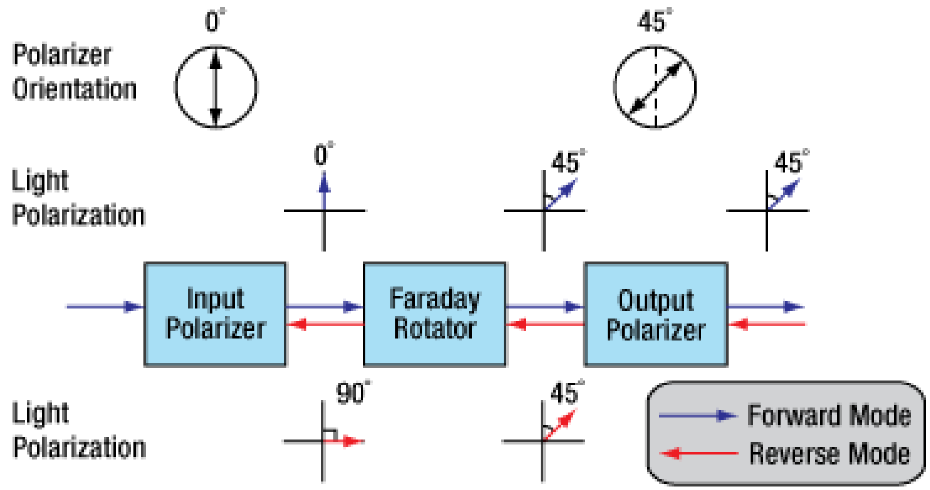
where $V$ is **Verdet constant**.

**Rotator Jones matrix:**

$$M = R(\theta) \qquad \theta = VBd$$

Because of proportionality to $B$, multi-pass configuration **adds** to the rotation.

## Faraday Effect Optical Isolator




**Exercise:** Construct the Jones matrix for the isolator

A) for the forward direction

B) for the backward direction

C) show that the device acts a (polarization-sensitive) isolator

**Hint:** Be careful about having the local frame of reference set up correctly for both directions

# Eigenvectors of Jones Matrices

## Eigenvector of M:

$$Mv = \lambda v$$

(Where $\lambda$ can also be zero.)
In the context of Jones calculus:

- eigenvectors of an optical-element matrix correspond to those polarization states that "do not change" but are "only" multiplied by a factor.
- those factors (eigenvalues) correspond to:
  — acquired (relative) phase change for wave plates
  — acquired (partial) phase change for polarizers, plus attenuation of the rejected polarization
- There are two eigenvectors for each element

**Example:** Quarter-wave plate

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

has the following polarization states as its eigenvectors:

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda = 1 \qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \lambda = i$$

**Note:** It is often easier to utilize such vectors as the basis of the polarization state vector space.

## Physical origin of Faraday effect

Susceptibility of a medium exposed to magnetic field (oriented along $z$):

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{P} = \epsilon_0 \bar{\chi}.\mathbf{E}$$

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

**Note:** Optically active media have the same type of susceptibility tensor, but in them $\chi_{12}$ does not depend on the external field!

**Exercise:**

A) Show that $\bar{\chi}$ preserves both left- and right-circularly polarized light, and that it **does not preserve linear polarization** upon propagation.

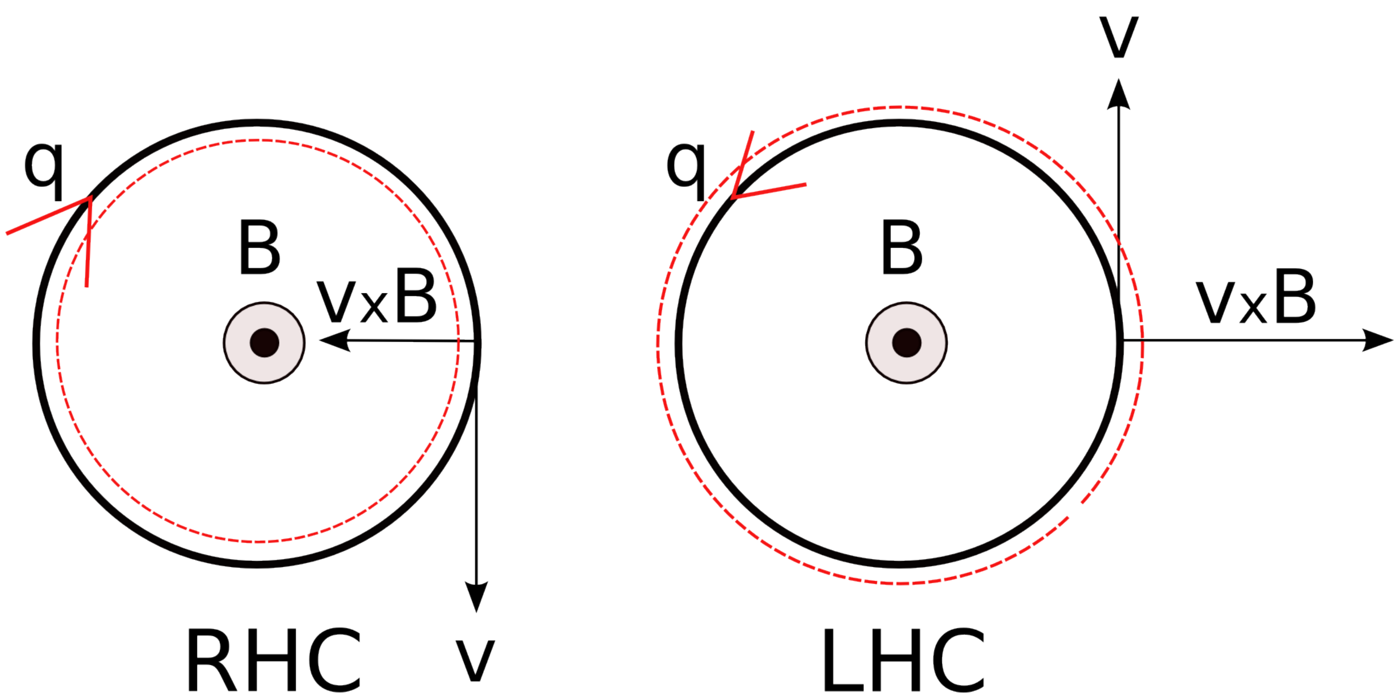
B) Calculate effective refractive index for RHC and LHC light. Express both in terms of $\chi_{ij}$, and show that

$$n_R = \sqrt{1 + \chi_{11} + \chi_{12}} \qquad n_L = \sqrt{1 + \chi_{11} - \chi_{12}}$$

C) Based on the previous result, relate the Verdet constant and the magnetic field $B$ to $\chi 12$. (see Fowles 6.9 for the method)

**Observation:** In a "Faraday medium," RHC and LHC polarization states are "eigenmodes" of propagation with different refractive indices.

## Physical origin of Faraday effect: Qualitative argument




- circularly polarized optical field causes electron to follow a circular "orbit"
- Lorentz force due to magnetic field is always normal to that orbit
- depending on the orientation, it tends to shrink or expand the orbit
- larger (smaller) orbit results in larger (smaller) polarization and concomitant modification of the index of refraction

## Spin angular momentum

Photon quantities:

- Energy per photon:

$$E = \hbar\omega$$

- Linear momentum:

$$\vec{p} = \hbar\vec{k}$$

- Angular momentum:

$$\vec{s} = \pm\hbar\hat{k}$$

Circularly polarized light also carries spin angular momentum associated with the rotating electric field.

Linear polarization is an equal-weight superposition of RHC and LHC and carries no spin angular momentum.