Light polarization:

- OPTI-310: fully polarized light
- Polarization state: 2D vector in a complex vector space
- Jones calculus: light = vector, optical element = matrix, system = string of matrices

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- Partially polarized light = polarized wave + natural light
- Unpolarized light = relative phases between vector components (of $\mathbf{E}$) change randomly, and faster than the detector can follow
- Polarization state: Stokes vector
- Muller matrix calculus

Isolating the polarization state of light (from an EM plane-wave)

Consider a harmonic plane-wave propagating along the z-axis

$$\vec{E}(\vec{r},t) = \hat{x}E_x(z,t) + \hat{y}E_y(z,t)$$

where

$$E_x(z,t) = A_x \cos(kz - \omega t + \phi_x) = \frac{1}{2}\left[\mathcal{E}_x e^{i(kz-\omega t)} + c.c.\right]$$

$$E_y(z,t) = A_y \cos(kz - \omega t + \phi_y) = \frac{1}{2}\left[\mathcal{E}_y e^{i(kz-\omega t)} + c.c.\right]$$

with *complex amplitudes*

$$\mathcal{E}_x = A_x e^{i\phi_x} \qquad \mathcal{E}_y = A_y e^{i\phi_y}$$

Q: If we could follow the “tip” of the electric field vector at a fixed location in space, over one period of the optical cycle, what (parametric) curve it will “draw” ?

Substitute (rename the phase argument that appears in the exponentials):

$$\tau = \omega t - kz$$

and get (using $\cos(a+b) = \cos a \cos b - \sin a \sin b$)

$$\frac{E_x(\tau)}{A_x} = \cos\tau\cos\phi_x + \sin\tau\sin\phi_x$$

$$\frac{E_y(\tau)}{A_y} = \cos\tau\cos\phi_y + \sin\tau\sin\phi_y$$

Manipulate to obtain **equation of ellipse**:

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\left(\frac{E_x}{A_x}\right)\left(\frac{E_y}{A_y}\right)\cos\phi_{yx} = \sin^2\phi_{yx}$$

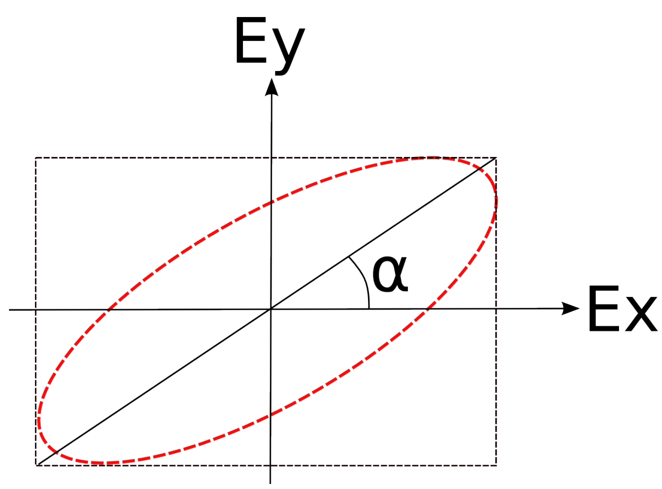
where

$$\phi_{yx} = \phi_y - \phi_x \equiv -\delta$$

A:

- Tip of the E-field vector in the $(x-y)$ plane must fall on the above ellipse parametrically with τ.
- orientation and ellipticity depend on A_x/A_y and ϕ_{yx}.

$$\tan 2\alpha = \frac{2\cos\phi_{yx} A_x A_y}{A_x^2 - A_y^2}$$



Example:

$\phi_{yx} = 0$ gives

$$\left(\frac{E_x}{A_x}\right) = \left(\frac{E_y}{A_y}\right)$$

or

$$E_y = \frac{A_y}{A_x} E_x$$

which represents a straight line of slope (A_y/A_x)

Example:

$\phi_{yx} = \pi$ gives the same, except the slope changes sign

For $A_x = A_y$, these are examples of **linear polarizations** with orientation of 45 and -45 degrees. Motion of the tip of $\vec{E}(\tau)$ is a straight line in the $(x - y)$ plane.

Summary so far:

- $\phi_{xy} = 0, \pm\pi$ implies linear polarization
- ratio A_x/A_y determines the orientation ("slope") of oscillation
- $A_y = 0$ means $\hat{x}$ polarization
- $A_x = 0$ means $\hat{y}$ polarization

Example:

$A_x = A_y$, $\phi_{yx} = +\pi/2$, circularly polarized, Left, LHC
$A_x = A_y$, $\phi_{yx} = -\pi/2$, circularly polarized, Right, RHC

Take the first case, $\phi_{yx} = +\pi/2$

$$\frac{E_x(\tau)}{A_x} = \cos\tau\cos\phi_x + \sin\tau\sin\phi_x = \cos\tau\cos\phi_x + \sin\tau\sin\phi_x$$

$$\frac{E_y(\tau)}{A_y} = \cos\tau\cos\phi_y + \sin\tau\sin\phi_y = -\cos\tau\sin\phi_x + \sin\tau\cos\phi_x$$

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\left(\frac{E_x}{A_x}\right)\left(\frac{E_y}{A_y}\right)\cos\phi_{yx} = \sin^2\phi_{yx}$$

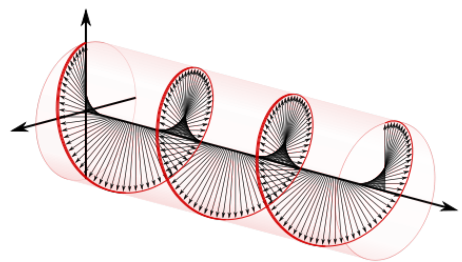
$$\left(\frac{E_x}{A}\right)^2 + \left(\frac{E_y}{A}\right)^2 - 2\left(\frac{E_x}{A}\right)\left(\frac{E_y}{A}\right)0 = 1$$

where

$$\phi_{yx} = \phi_y - \phi_x = \pi/2$$

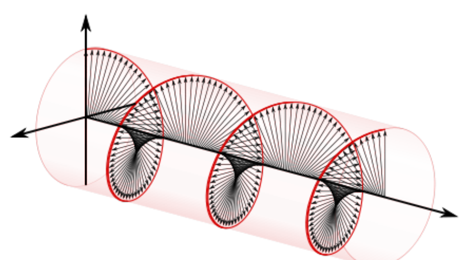
Right polarized light:

- clock-wise rotation (with time) at a given point in space when viewed against direction of propagation
- for a fixed time (a snapshot), vectors describe right-handed spiral



Left polarized light:

- anti-clock-wise rotation (with time) at a given point in space when viewed against direction of propagation
- for a fixed time (a snapshot), vectors describe left-handed spiral



Complex-valued polarization vectors:

Take

$$\vec{E}(\vec{r},t) = A\left[\hat{x}\cos(kz-\omega t) + \hat{y}\sin(kz-\omega t)\right] = A\left[\hat{x}\cos\tau - \hat{y}\sin\tau\right]$$

which is RHC light (Why?, Keep in mind that $\tau = \omega t - kz$).

$$\vec{E}(\vec{r},t) = \frac{A}{2}\left[\hat{x}(e^{i\tau}+e^{-i\tau}) - \hat{y}(e^{i\tau}-e^{-i\tau})/i\right] = \frac{A}{\sqrt{2}}\left[\frac{\hat{x}+i\hat{y}}{\sqrt{2}}e^{i\tau} + \frac{\hat{x}-i\hat{y}}{\sqrt{2}}e^{-i\tau}\right]$$

$$\vec{E}(\vec{r},t) = \frac{A}{\sqrt{2}}\left[\frac{\hat{x}+i\hat{y}}{\sqrt{2}}e^{-i(kz-\omega t)} + \frac{\hat{x}-i\hat{y}}{\sqrt{2}}e^{+i(kz-\omega t)}\right]$$

Normalized, complex-valued, polarization vector:

$$\hat{e}_{\pm} = \frac{1}{\sqrt{2}}(\hat{x} \mp i\hat{y})$$

RHC (+) and LHC (-) constitute a basis. They are also orthogonal:

$$\hat{e}_\mu . \hat{e}^*_\nu = \delta_{\mu\nu}$$

In general: The are two linearly independent polarization states for any given direction of propagation (here z). They form and orthonormal basis.

Note: keep in mind that the scalar product is now "complex"!

Example: $\phi_{yx} \neq 0, \pm\pi$, elliptical polarization states

- $0 < \phi_{yx} < \pi$ is left-handed elliptic polarization
- $\pi < \phi_{yx} < 2\pi$ is right-handed elliptic polarization

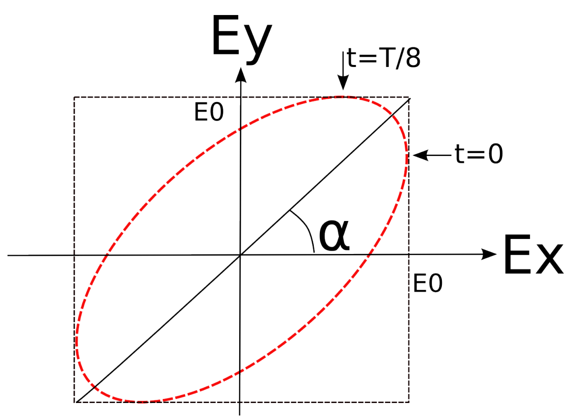
$$\vec{E}(z,t) = \hat{\imath} E_0 \cos(kz - \omega t) + \hat{\jmath} E_0 \cos(kz - \omega t + \pi/4)$$

What is the polarization state of this wave?

$$\vec{E}(z=0, t=0) = \hat{\imath} E_0 \cos(0) + \hat{\jmath} E_0 \cos(0 + \pi/4) = \hat{\imath} E_0 + \hat{\jmath} E_0/\sqrt{2}$$

$$\vec{E}(z=0, t=T/8) = \hat{\imath} E_0 \cos(-\pi/4) + \hat{\jmath} E_0 \cos(-\pi/4 + \pi/4) = \hat{\imath} E_0/\sqrt{2} + \hat{\jmath} E_0$$

$$\vec{E}(z=0, t=T/4) = \hat{\imath} E_0 \cos(-\pi/2) + \hat{\jmath} E_0 \cos(-\pi/2 + \pi/4) = \hat{\imath} 0 + \hat{\jmath} E_0/\sqrt{2}$$



What is the orientation of the polarization ellipse?

$$\tan 2\alpha = \frac{2E_0^2}{E_0^2 - E_0^2}\cos(\pi/4) = \infty \qquad \alpha = 45^o$$

What is the maximum and minimum field amplitude?

A) Maximum occurs halfway between $t = 0$ and $t = T/8$, i.e. at $t = T/16$, so

$$\vec{E}(z = 0, t = T/16) = \hat{\imath}E_0 \cos(-2\pi/16) + \hat{\jmath}E_0 \cos(-\pi/8 + \pi/4)$$

$$(\vec{E}.\vec{E}) = 2E_0^2 \cos^2(\pi/8) \qquad E_{max} = \sqrt{2}\cos(\pi/8)E_0 = 1.31E_0$$

B) Minimum one quarter of a cycle later, at $t = T/16 + T/4$ i.e. at $t = 5T/16$, so

$$\vec{E}(z = 0, t = 5T/16) = \hat{\imath}E_0 \cos(-5\pi/8) + \hat{\jmath}E_0 \cos(-5\pi/8 + \pi/4)$$

$$\vec{E}(z = 0, t = 5T/16) = \hat{\imath}E_0 \cos(-5\pi/8) + \hat{\jmath}E_0 \cos(-3\pi/8) = -\hat{\imath}E_0 \sin(\pi/8) + \hat{\jmath}E_0 \sin(\pi/8) =$$

$$(\vec{E}.\vec{E}) = 2E_0^2 \sin^2(\pi/8) \qquad E_{min} = \sqrt{2}\sin(\pi/8)E_0 = 0.54E_0$$

Example:

$$\begin{aligned} E_x(z,t) &= \hat{i}E_{0x}\cos(kz-\omega t) \\ E_y(z,t) &= \hat{j}E_{0y}\cos(kz-\omega t+\pi/2) \end{aligned}$$

A) What is the polarization state of this wave?

B) What can you say about the orientation of the polarization ellipse?

C) Under what condition is this a circularly polarized wave?

Answer:

A) Since in general we have no specification of amplitudes $E_{0x,y}$ it is safe to say that in most cases this will be an elliptically polarized wave. Note that without being sure about the signs of the two amplitudes, we can't even tell if the wave is right- or left-polarized.

B) In view of the above remark, we assume that that $E_{0x,y}$ are both positive. This is reasonable because any negative sign could have been "absorbed" in the appropriate phase of cos functions.

First, identify a point corresponding to the tip of the E-vector for $z = 0, t = 0$:

$$\vec{E}(z = 0, t = 0) = E_{0x}\hat{i}$$

Second, figure out which way vector $\vec{E}$ departs from this point as as time increases, so that $t > 0$, but small:

$$E_y(z = 0, t > 0) \sim E_{0y} \cos(-\omega t + \pi/2) > 0$$

so the vector departs in the positive vertical direction, and that means that the "rotation" will be counter-clock-wise. This is therefore left-handed polarization. We can also see that the ellipse is oriented horizontally (or vertically).

C) Obviously, the bounding box for circularly polarized wave must be a square, and we need to ask for $E_{0x} = E_{0y}$.

Jones Calculus: Polarization Vectors

State of polarization represented by *complex* amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ (assuming propagation along z). Relation to the polarization ellipse:

$$\phi_{yx} = \arg(\mathcal{E}_x/\mathcal{E}_y) \qquad\qquad (A_x/A_y) = \left|\frac{\mathcal{E}_x}{\mathcal{E}_y}\right|$$

Jones Vector:

$$\bar{\mathcal{E}}(z) = \begin{pmatrix} \mathcal{E}_x(z) \\ \mathcal{E}_y(z) \end{pmatrix}$$

A complex-valued, two-component column vector:

- Usually normalized to unity, but not always
- It is the **ratio of components** that is important
- Can factor-out arbitrary complex factor
- In problems dealing with transmitted power, normalization **is important**
- Power in the beam = norm^2 of the Jones vector:

$$< \bar{\mathcal{E}}|\bar{\mathcal{E}} >= (\bar{\mathcal{E}}^T)^*.\bar{\mathcal{E}} = \bar{\mathcal{E}}^+.\bar{\mathcal{E}} = \mathcal{E}_x^*\mathcal{E}_x + \mathcal{E}_y^*\mathcal{E}_y = |\mathcal{E}_x|^2 + |\mathcal{E}_y|^2$$

Jones Calculus: Polarization Vectors

- x-polarized:

$$\bar{\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- y-polarized:

$$\bar{\mathcal{E}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- RHC, right circular polarization:

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- LHC, left circular polarization:

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}$$

- linear polarization, +45 deg

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- linear polarization, −45 deg

$$\bar{\mathcal{E}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Jones Calculus: Polarization Vectors and their transformation by optical elements

Note: If the problem is linear, the relation between Jones vector before and after transition through an optical element must be also linear:

$$\begin{pmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{pmatrix}_{after} = M \begin{pmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{pmatrix}_{before}$$

where M is some 2×2 (complex-valued) matrix.

This obviously generalizes to a system consisting of a number of elements:

$$\begin{pmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{pmatrix}_{after} = M_n \dots M_i \dots M_1 \begin{pmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{pmatrix}_{before}$$

Note: The first matrix in the string is for the element encountered first.

Jones Calculus: Matrices for Optical Elements

Polarizers:

- Linear polarizer, transmission horizontal (x)

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Linear polarizer, transmission vertical (y)

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Linear polarizer, transmission axis at ± 45 deg

$$M = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

- Circular polarizer, Right and Left:

$$M = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad M = \frac{1}{2} \begin{pmatrix} 1 & -i \\ +i & 1 \end{pmatrix}$$

Note: Polarizer matrices have zero determinants. Why?

Jones Calculus: Matrices for Optical Elements

Plates:

- Quarter wave-plate, fast axis vertical:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

- Quarter wave-plate, fast axis horizontal:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & +i \end{pmatrix}$$

- Quarter wave-plate, fast at ± 45 deg

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$$

- Half wave-plate, fast axis horizontal or vertical

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: Plates have $|\det M| = 1$, i.e. they are *unitary* operators.

Rotated optical elements (and their Jones matrices):

Consider a Jones vector

$$\bar{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

in a rotated basis system $\hat{e}'_x, \hat{e}'_y$

$$\begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix} = R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^{-1}(-\theta)$$

Take

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_{z=L} = M \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_{z=0}$$

and represent the same in the rotated frame of reference:

$$\begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix}_L = R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_L = R(\theta) M \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_0 = R(\theta) M R^{-1}(\theta) R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_0 \equiv M' R(\theta) \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}_0 = M' \begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix}_0$$

so the Jones matrix in the rotated system is

$$M' = R(\theta) M R^{-1}(\theta)$$

Rotated optical elements (and their Jones matrices):

Assume it is the element M' in the rotated system that we know. Then back in the lab frame we have:

$$M = R^{-1}(\theta) M' R^(\theta) = R(-\theta) M' R^(\theta)$$

Example: Rotated polarizer
In the rotated frame it is

$$M' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

What is it in the lab frame:

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

Example: Consider a LHC polarized wave incident on a rotated polarizer:

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} \mathcal{E}_x(0) \\ \mathcal{E}_y(0) \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} \mathcal{E}_x(L) \\ \mathcal{E}_y(L) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2\theta + i\sin\theta\cos\theta \\ \sin\theta\cos\theta + i\sin^2\theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta e^{i\theta} \\ \sin\theta e^{i\theta} \end{pmatrix} = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

This means that

$$\phi_{yx}(L) = \arg(\cot\theta) = 0, \pi$$

$$\left(\frac{A_y}{A_x}\right) = \tan\theta$$

So the action is: Circular $\rightarrow$ Linear

Q: Do this for "any" incident polarization state and show that the result is "the same" with one exception.