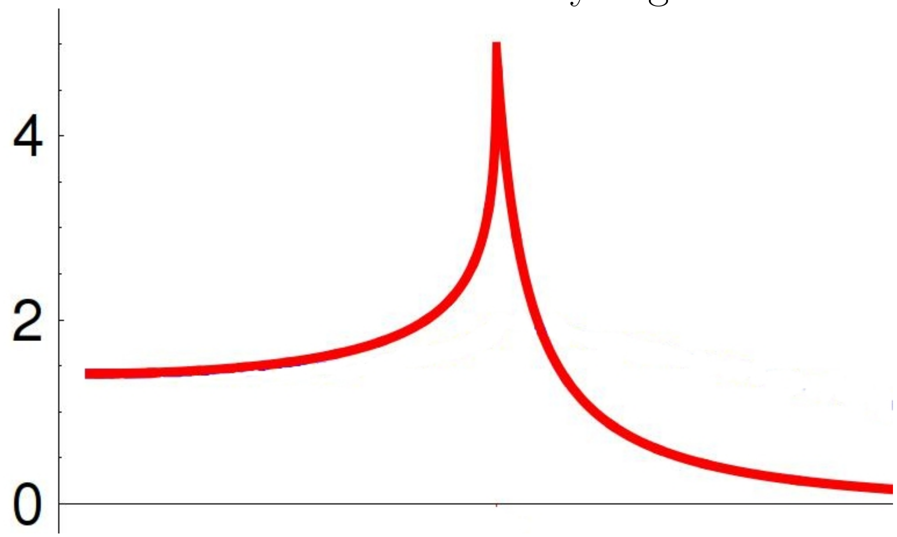
This material contains qualitative practice problems concerning Fresnel formulas. It is OK to use a cheat sheet, as the idea is to identify important features in how reflection and transmission depends on various parameters, rather than memorizing equations.

Fresnel puzzler:

The following picture shows one of the Fresnel-formulae related quantities r_s, r_p, t_s, t_p, R_s, R_p, T_s, T_p as a function of the incidence angle. Only the vertical scale is specified, while the horizontal may or may not be shown over the full interval of zero to ninety degrees.

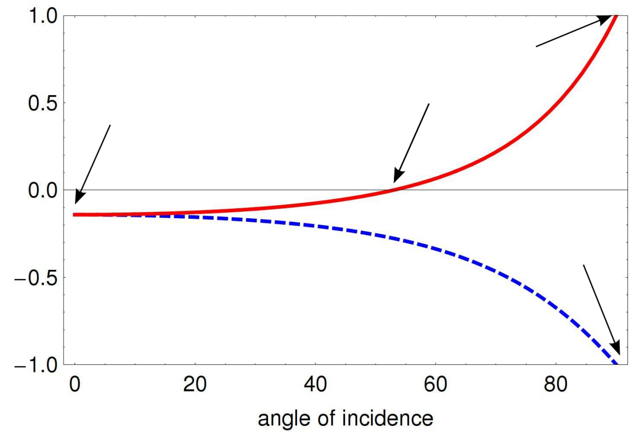


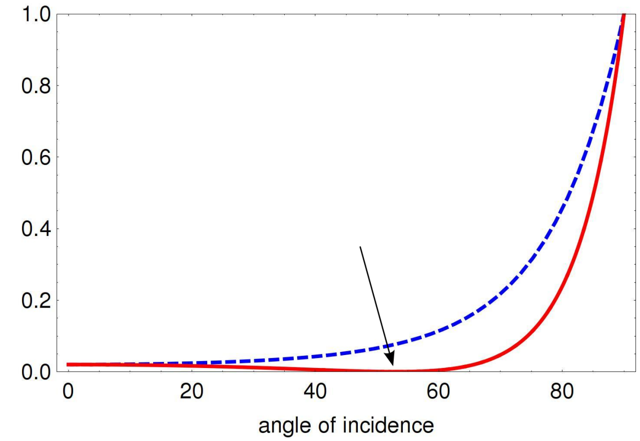
A Which quantity is shown?

B What is the relative refractive index of the interface?

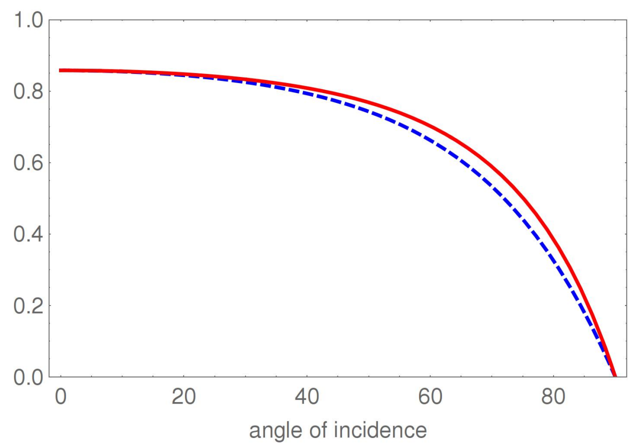
C Is this for internal or external incidence?

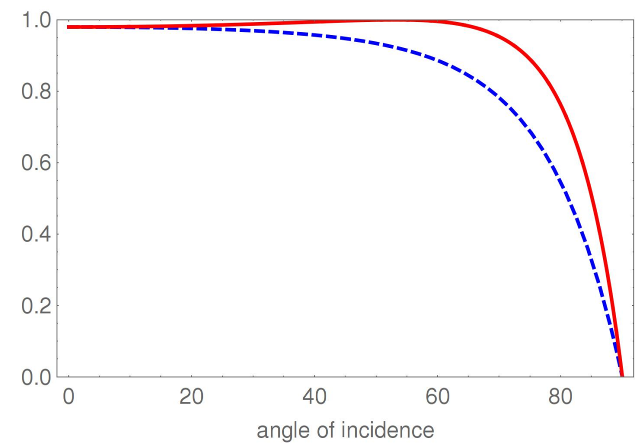
reflection coeffs. water, external reflectance





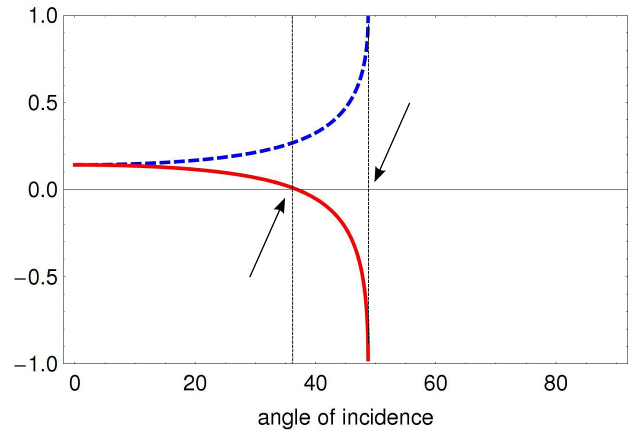
transmission coeffs. water, external transmittance

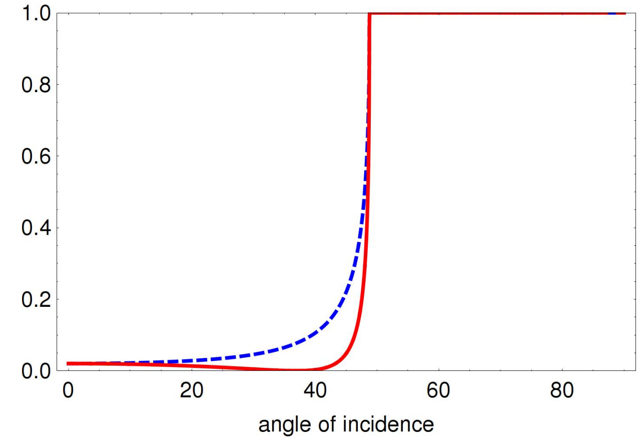




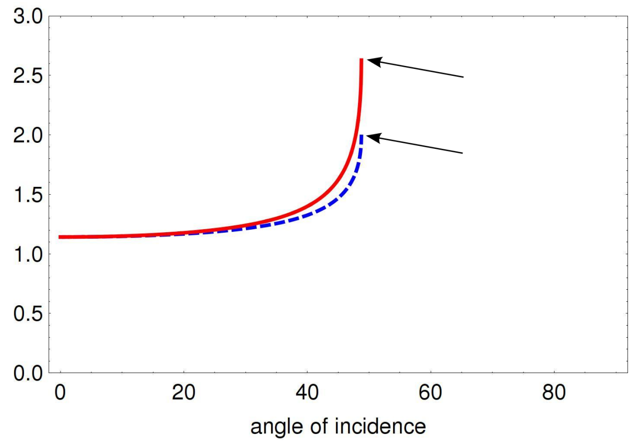
$$r_s = \frac{\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_s = \frac{2\cos\Theta_i}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad r_p = -\frac{n^2\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_p = \frac{2n\cos\Theta_i}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

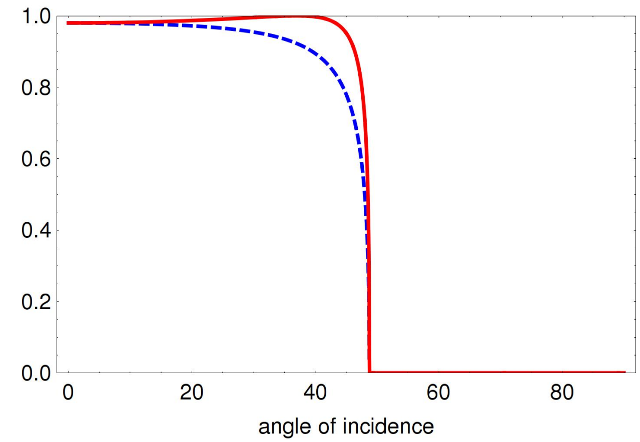
reflection coeffs. water, internal reflectance





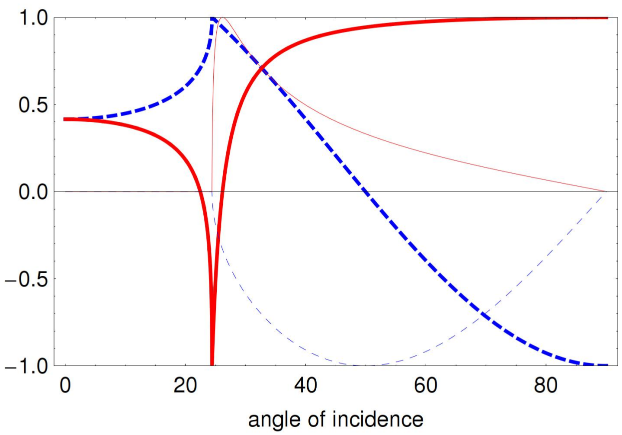
transmission coeffs. transmittance

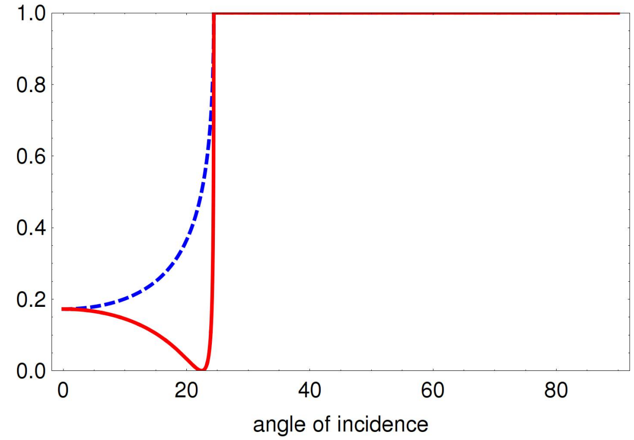




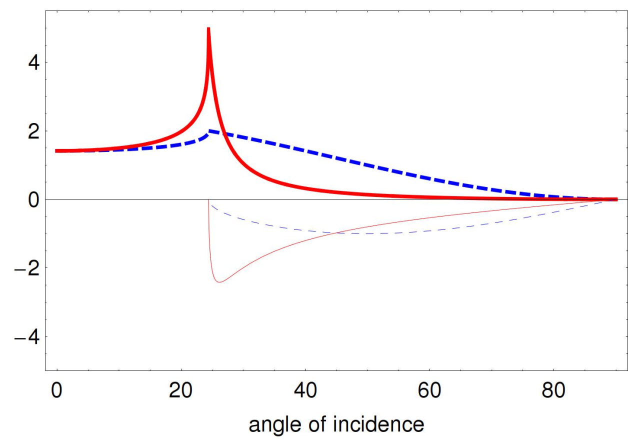
$$r_s = \frac{\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_s = \frac{2\cos\Theta_i}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad r_p = -\frac{n^2\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_p = \frac{2n\cos\Theta_i}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

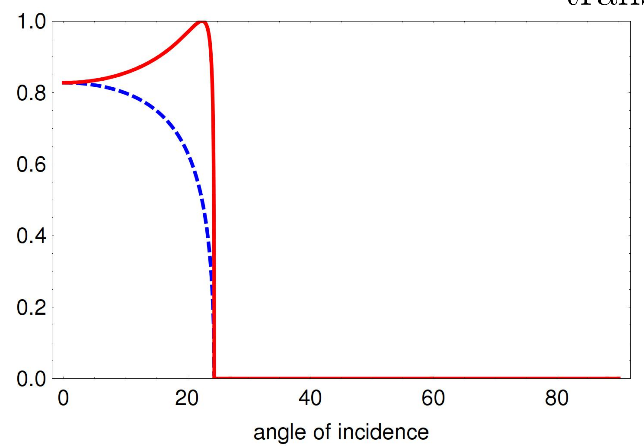
reflection coeffs. diamond, internal reflectance





transmission coeffs. transmittance



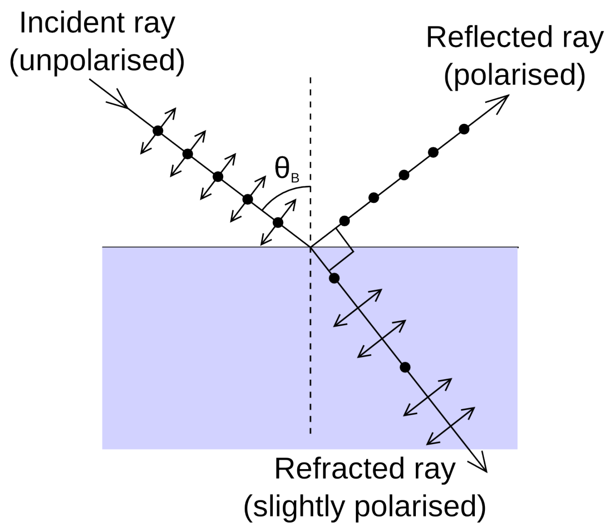


$$r_s = \frac{\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_s = \frac{2\cos\Theta_i}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad r_p = -\frac{n^2\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_p = \frac{2n\cos\Theta_i}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

Brewster angle, windows

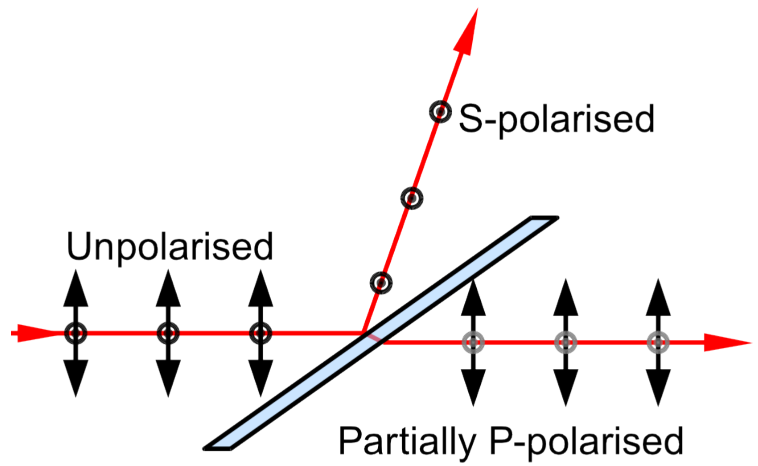
We have seen that there is an angle for which the TM reflectivity coefficient vanishes. Find the value of $\theta_{Brewster}$:

$$r_p(\theta_B) = 0 \Rightarrow \tan\theta_B = n$$



- no reflection for TM
- reflected wave is S-polarized
- θ_i and θ_t are complementary
- reflected and refracted rays are perpendicular

Brewster angle, windows



- perfect windows for TM
- imperfect window for TE
- can be used to impart losses on unwanted polarization

Exercise:
Consider a plan-parallel slab of refractive index n. Show that if the angle of incidence is Brewster, then the Brewster-angle condition is also fulfilled when the ray exits through the second interface. In other words, TM polarization transmits without losses (and reflections) through both interfaces of a Brewster window.

Evanescent waves in Total Internal Reflection (TIR)

All the formulas we have derived (Fresnel and conditions on the wave-vector due to translation invariance of the interface) remain valid for incidence angles beyond critical — the "only" difference is that they become complex-valued.

In particular

$$\vec{k}'' = \vec{k}_{\parallel} + \vec{k}''_{\perp}$$

still holds. Of course it must satisfy the dispersion relation, i.e.

$$(k'')^2 = \frac{\omega^2 n_t^2}{c^2}$$

and this means that

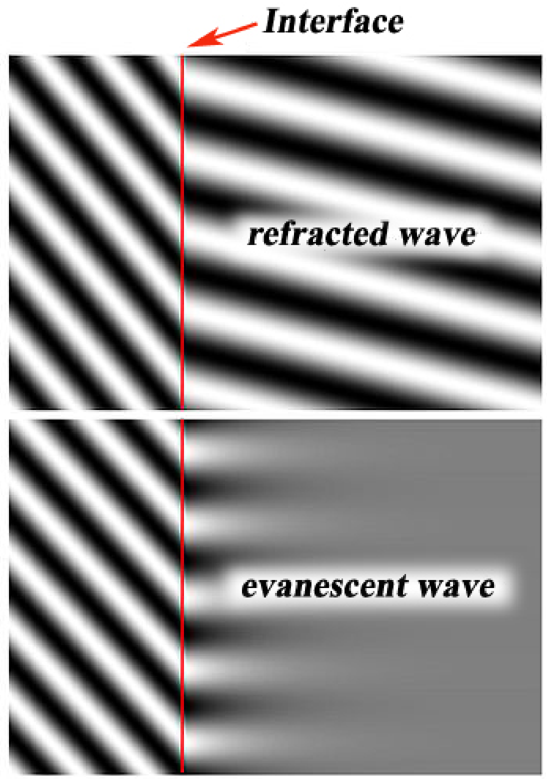
$$k''_{\perp} = \sqrt{\frac{\omega^2 n_t^2}{c^2} - k_{\parallel}^2} = \sqrt{\frac{\omega^2 n_t^2}{c^2} - \frac{\omega^2 n_i^2}{c^2}\sin^2\theta_i} = \frac{\omega n_i}{c}\sqrt{\frac{n_t^2}{n_i^2} - \sin^2\theta_i} = -\frac{i\omega n_i}{c}\sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}$$

because for angles beyond critical, $\sin\theta_i > n$. **Thus, the normal component of the transmitted wave-vector is purely imaginary.**

Note: Its sign must be chosen such that the resulting "refracted" wave will decay at large (in comparison to λ) distances from the interface:

$$\{\vec{E}'', \vec{H}''\} \exp\left[i(\vec{k}''.\vec{r} - \omega t)\right] = \{\vec{E}'', \vec{H}''\} \exp\left[i(x\frac{\omega n_i}{c}\sin\theta_i - \omega t)\right] \exp\left[y\frac{\omega n_i}{c}\sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}\right]$$

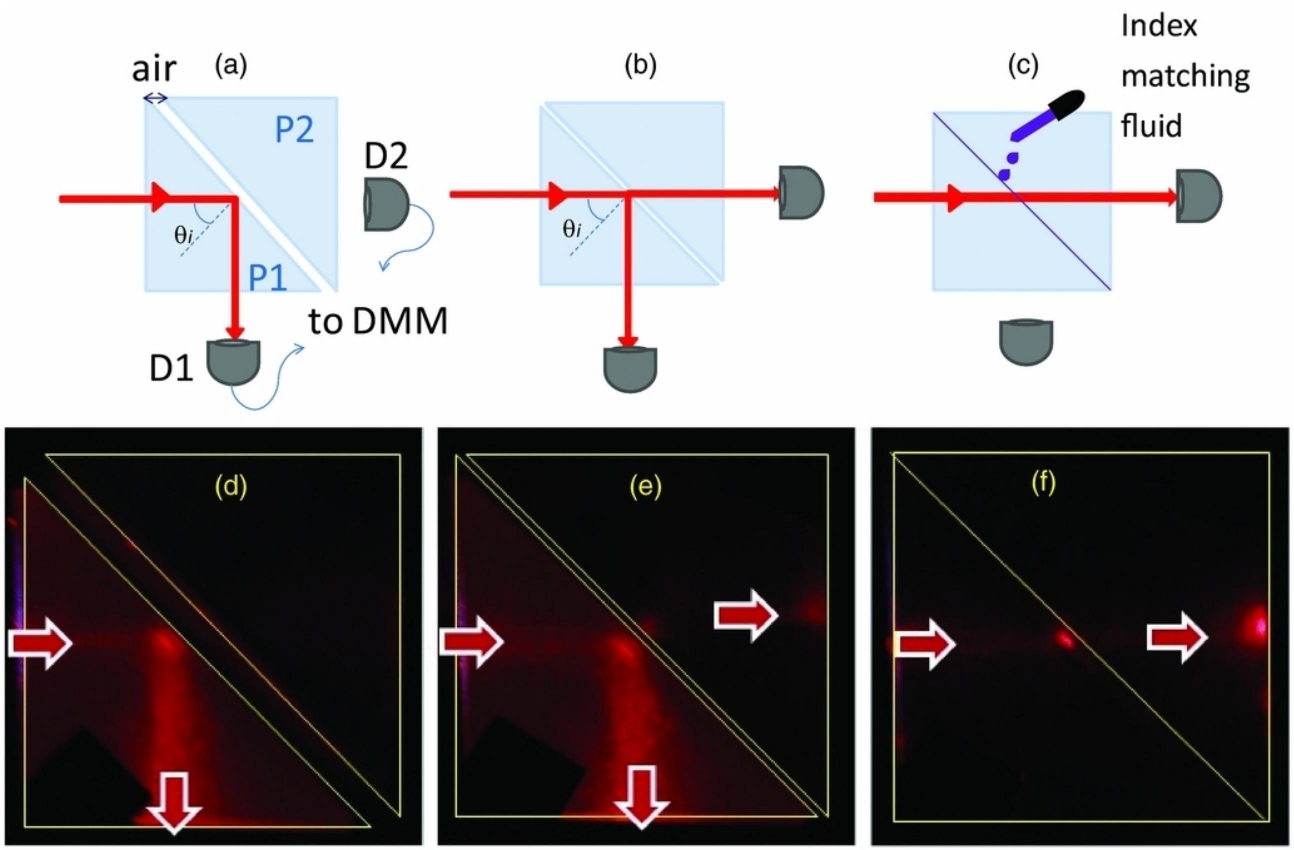
Evanescent waves in Total Internal Reflection (TIR)



$$\{\vec{E}'', \vec{H}''\} \exp\left[i(\vec{k}''.\vec{r} - \omega t)\right] = \{\vec{E}'', \vec{H}''\} \exp\left[i(x\frac{\omega n_i}{c}\sin\theta_i - \omega t)\right] \exp\left[y\frac{\omega n_i}{c}\sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}\right]$$

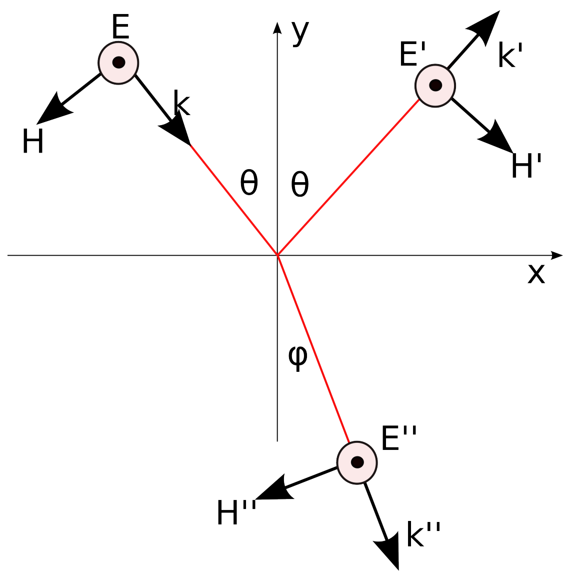
Frustrated TIR:

If the evanescent field extends to a nearby higher-index medium, it "converts" into a propagating wave, and thus mediates non-zero transmission and decreases the reflectivity:



R. Salman et al. Eur. J. Phys. 34(2013)1439.

Poynting vector in the evanescent field



$$\vec{E} = \hat{z}E'' \exp\left[i(x\frac{\omega n_i}{c}\sin\theta_i - \omega t)\right] \exp\left[y\frac{\omega n_i}{c}\sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}\right] = \hat{z}E'' e^{ik''_x x} e^{\alpha y} e^{-i\omega t}$$

Need magnetic field:

$$\vec{H} = \frac{1}{i\omega\mu_0}\nabla \times \vec{E} = \frac{1}{i\omega\mu_0}\left[\alpha\hat{x} - ik''_x\hat{y}\right] E'' e^{ik''_x x} e^{\alpha y} e^{-i\omega t}$$

Now calculate Poynting:

$$\vec{S} = \frac{1}{4}\left(\vec{E} + \vec{E}^*\right) \times \left(\vec{H} + \vec{H}^*\right)$$

$$\langle\vec{S}\rangle_T = \frac{1}{4}\left(\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}\right)$$

Poynting vector in the evanescent field

$$\vec{E} \times \vec{H}^* = \left[\hat{z} E'' e^{ik''_x x} e^{\alpha y} e^{-i\omega t}\right] \times \left[\frac{-1}{i\omega\mu_0}\left[\alpha\hat{x} + ik''_x\hat{y}\right](E'')^* e^{-ik''_x x} e^{\alpha y} e^{+i\omega t}\right]$$

$$\vec{E} \times \vec{H}^* = -\frac{|E''|^2 e^{2\alpha y}}{i\omega\mu_0}\left[\alpha\hat{y} - ik''_x\hat{x}\right]$$

$$\langle\vec{S}\rangle_T = \frac{1}{2}\frac{k''_x}{\omega\mu_0} e^{2\alpha y}|E''|^2\hat{x}$$

- Time-averaged Poynting has a component tangential to the interface
- No steady flow of energy perpendicular to the interface
- The decay of the field, and the extent of the spatial region with the tangential energy flow, is controlled by (keep in mind that $y < 0$):

$$e^{2\alpha y} \quad , \quad \alpha = \frac{\omega n_i}{c}\sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}$$

  where $1/\alpha$ plays a role of a “depth” length scale.
- The “depth” is diverging as θ_i approaches the critical angle value.
- Away from critical, the “depth” is on the order of a wavelength