Interface conditions summary:

A) Component form (the form we use in OPTI-310, remember this) :

$$\begin{aligned} D_{n1} - D_{n2} &= 0 \\ B_{n1} - B_{n2} &= 0 \\ E_{t1} - E_{t2} &= 0 \\ H_{t1} - H_{t2} &= 0 \end{aligned}$$

B) In words:
1. Normal components of B and D are continuous at the material interface
2. Tangential components of E and H are continuous at the material interface

Note: Normal and tangential is meant with respect to the normal vector of the interface

Dielectric Interface

Situation:

- Planar interface between two dielectrics
- EM plane wave incident
- Produces refracted and reflected waves (symmetry dictates that they are also plane waves)
- We will use material-interface conditions for fields to figure out their properties

book: Fowles p. 38-51

Law of reflection and refraction

Incident wave:

$$\vec{E} \exp\left[i(\vec{k}.\vec{r} - \omega t)\right] \qquad \vec{H} \exp\left[i(\vec{k}.\vec{r} - \omega t)\right]$$

Reflected wave:

$$\vec{E}' \exp\left[i(\vec{k}'.\vec{r} - \omega t)\right] \qquad \vec{H}' \exp\left[i(\vec{k}'.\vec{r} - \omega t)\right]$$

Refracted wave:

$$\vec{E}'' \exp\left[i(\vec{k}''.\vec{r} - \omega t)\right] \qquad \vec{H}'' \exp\left[i(\vec{k}''.\vec{r} - \omega t)\right]$$

Continuity of tangential field components dictates:
The phases of the three waves must agree at every point along the interface:

$$\vec{k}.\vec{r}_b = \vec{k}'.\vec{r}_b = \vec{k}''.\vec{r}_b$$

which also means that for arbitrary $\vec{r}_1$, $\vec{r}_2$ (both at the boundary) we must have:

$$\vec{k}.(\vec{r}_1 - \vec{r}_2) = \vec{k}'.(\vec{r}_1 - \vec{r}_2) = \vec{k}''.(\vec{r}_1 - \vec{r}_2)$$

... and this we read as:
1. Tangential components of the three wave-vectors must be the same

Next, because $\vec{k}$ and $\vec{k}'$ satisfy the same dispersion relation (same medium!),
2. The normal component of the incident and reflected wave-vectors must have the same magnitudeg

Translate 1. and 2. into relation between wave-vectors:

$$\vec{k} = \vec{k}_{||} + \vec{k}_{\perp}$$

$$\vec{k}' = \vec{k}_{||} - \vec{k}_{\perp}$$

$$\vec{k}'' = \vec{k}_{||} - \vec{k}''_{\perp}$$

Let θ, θ' and ϕ are the angles between wave-vectors and the normal to the interface.

From 1. and 2. applied to $\vec{k}$ and $\vec{k}'$, we conclude that

$$\theta' = \theta$$

i.e. angle of reflection is equal to the angle of incidence

From 1. applied to $\vec{k}$ and $\vec{k}''$, we conclude that

$$k \sin\theta = k'' \sin\phi \ ,$$

or

$$\frac{\omega n_1}{c} \sin\theta = \frac{\omega n_2}{c} \sin\phi \ ,$$

or

$$n_1 \sin\theta = n_2 \sin\phi \ ,$$

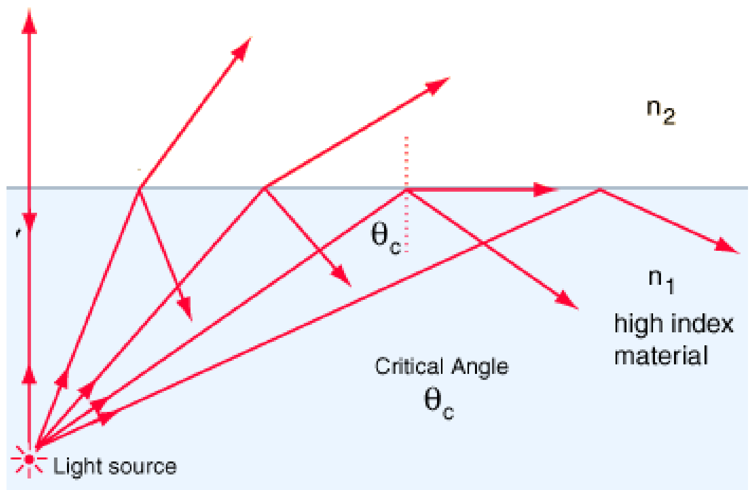
which is the **Snell's law of refraction**

Sometimes expressed with the *relative index*

$$\frac{\sin\theta}{\sin\phi} = n \equiv \frac{n_2}{n_1}$$

Critical angle and Total Internal Reflection (TIR):

- light propagates in the optically denser medium ($n_1 > n_2$)
- angle of incidence must be larger than the **critical angle**
- beyond the critical angle of incidence, there is no real-valued solution for the angle of refraction



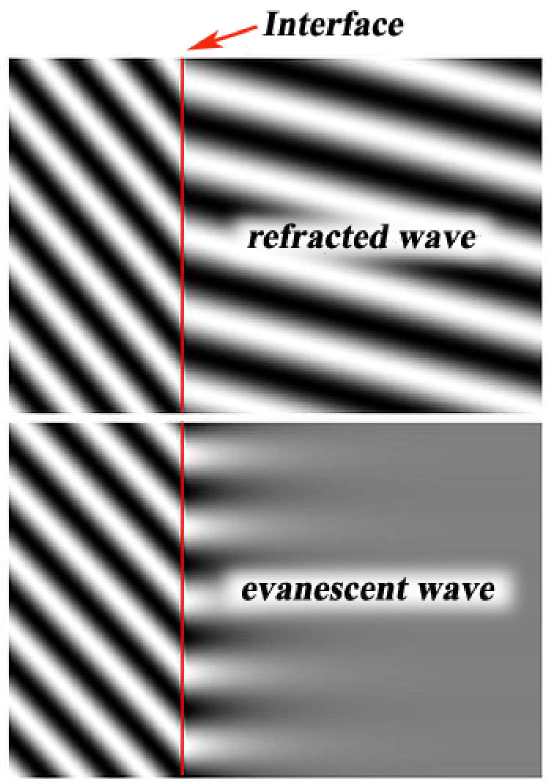
$$\sin\phi = \frac{n_1}{n_2}\sin\theta$$

RHS may become larger than one if θ is sufficiently large **and** $n_1/n_2 > 1$. The critical angle θ_c satisfies:

$$1 = \sin\frac{\pi}{2} = \sin\phi = \frac{n_1}{n_2}\sin\theta_c \qquad \theta_c = \arcsin\frac{n_2}{n_1}$$

Evanescent wave:

Q: What happens to the optical field in the "forbidden territory" behind the interface due to TIR?



Example: Calculate the critical angle for the diamond-air interface.

$$\theta_c = \arcsin\frac{n_2}{n_1} = \arcsin\frac{1}{2.42} = 24.4^o$$

Reflected and refracted fields:

What we aim to do: Solve Helmholtz equation in each medium separately, and match them to satisfy boundary conditions.

Incident (E and H fields):

$$\vec{E}_i(\vec{r},t) = \vec{E}\exp\left[i(\vec{k}.\vec{r} - \omega t)\right] \qquad \vec{H}_i(\vec{r},t) = \frac{1}{\mu\omega}\vec{k}\times\vec{E}\exp\left[i(\vec{k}.\vec{r} - \omega t)\right]$$

Reflected:

$$\vec{E}_r(\vec{r},t) = \vec{E}'\exp\left[i(\vec{k}'.\vec{r} - \omega t)\right] \qquad \vec{H}_r(\vec{r},t) = \frac{1}{\mu\omega}\vec{k}'\times\vec{E}'\exp\left[i(\vec{k}'.\vec{r} - \omega t)\right]$$

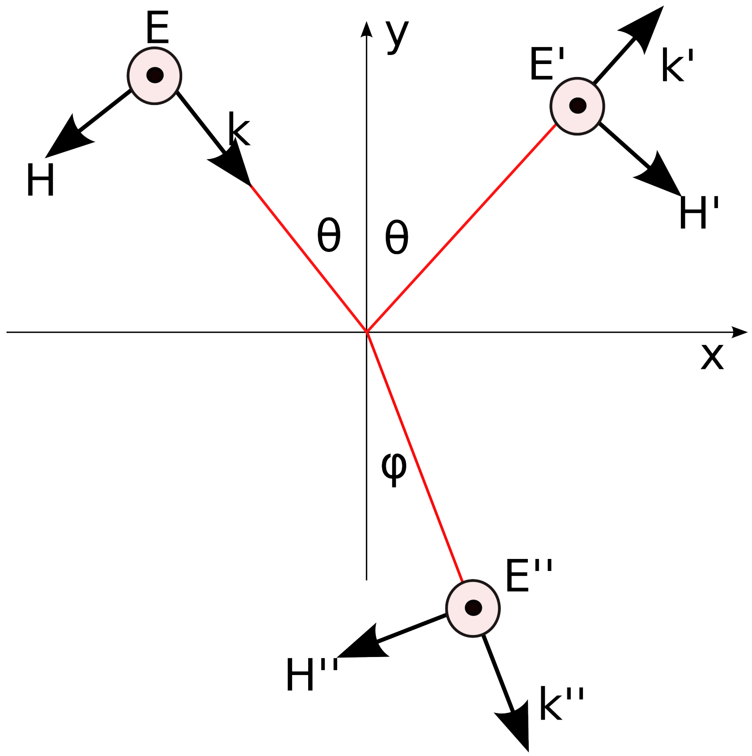
Refracted (transmitted):

$$\vec{E}_t(\vec{r},t) = \vec{E}''\exp\left[i(\vec{k}''.\vec{r} - \omega t)\right] \qquad \vec{H}_t(\vec{r},t) = \frac{1}{\mu\omega}\vec{k}''\times\vec{E}''\exp\left[i(\vec{k}''.\vec{r} - \omega t)\right]$$

We split the whole treatment into two cases:

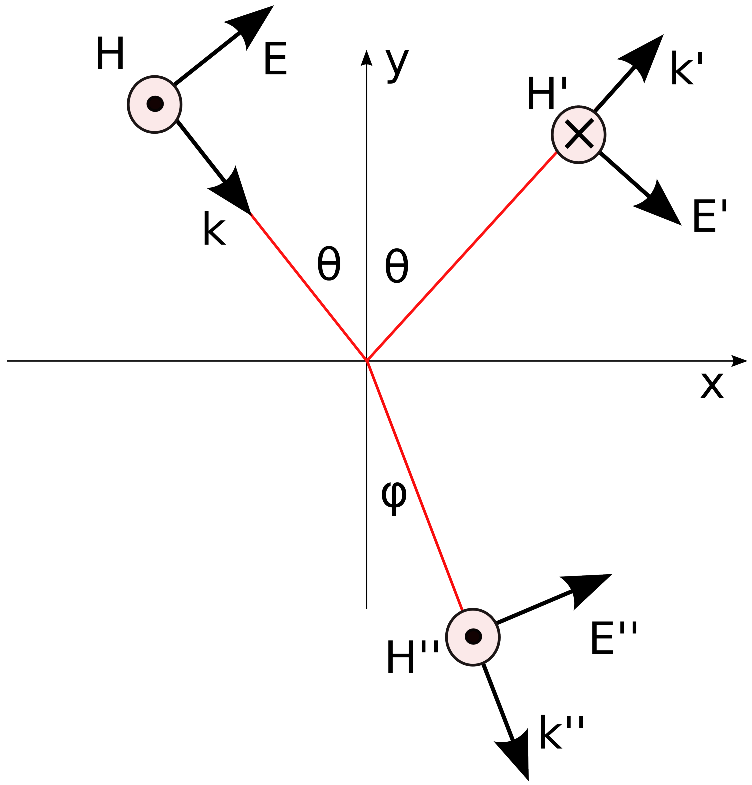
- TE, transverse electric, s-polarized
- TM, transverse magnetic, p-polarized

S-polarized case



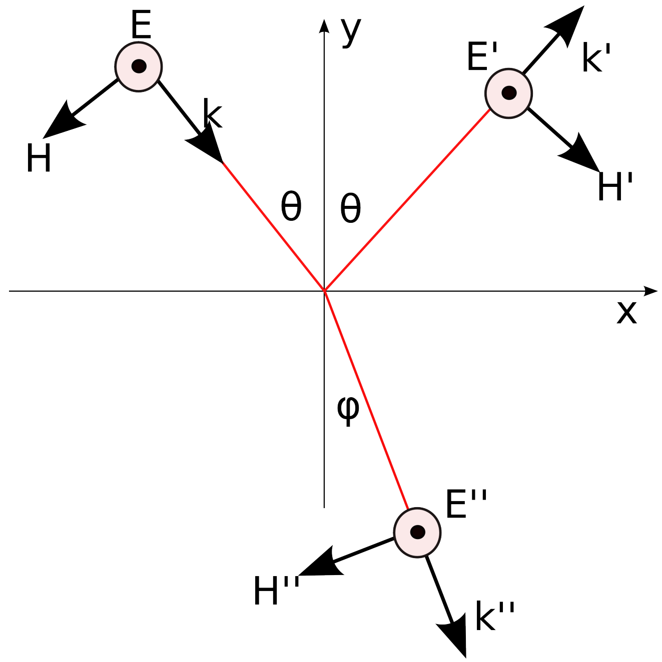
Transverse electric (TE): $\vec{E} \perp$ to plane of incidence

P-polarized case



Transverse magnetic (TM): $\vec{H} \perp$ to plane of incidence

S-polarized case: reflection and transmission coefficients



Require continuity of tangential components of electric and magnetic intensity:

$$-H\cos\theta + H'\cos\theta = -H''\cos\phi$$

... and express every H in terms of E:

$$-kE\cos\theta + k'E'\cos\theta = -k''E''\cos\phi$$

$$E + E' = E''$$

This is a system of equations for E' and E'' for a given E. Solve it...

S-polarized case: reflection and transmission coefficients

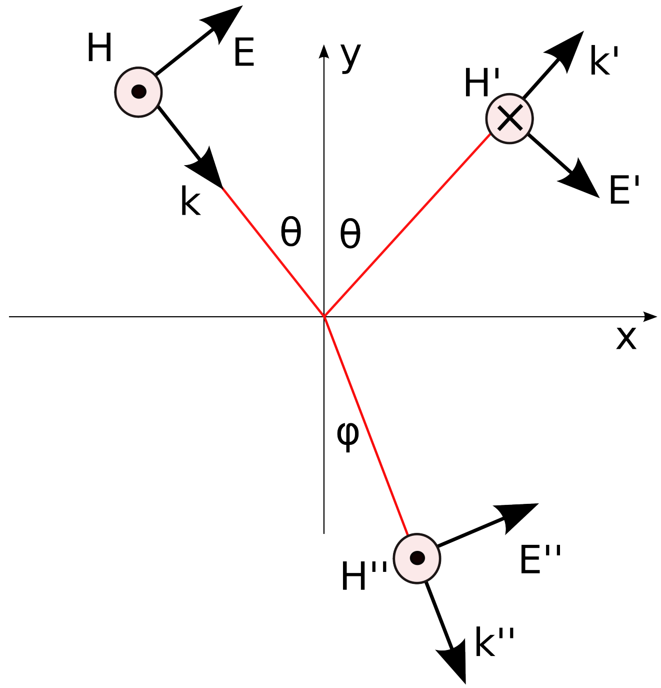
$$r_\perp \equiv r_s = \left[\frac{E'}{E}\right]_{TE} \qquad t_\perp \equiv t_s = \left[\frac{E''}{E}\right]_{TE}$$

$$r_s = \frac{n_1 \cos\theta - n_2 \cos\phi}{n_1 \cos\theta + n_2 \cos\phi} \quad , \quad t_s = \frac{2n_1 \cos\theta}{n_1 \cos\theta + n_2 \cos\phi}$$

$$r_\perp = \frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} \quad , \quad t_\perp = \frac{2n_i \cos\Theta_i}{n_i \cos\Theta_i + n_t \cos\Theta_t}$$

Note: both subscripts, s and $\perp$, are often used to indicate the S-polarized case. Also, be ready to see different symbols for incident and transmitted angles.

P-polarized case: reflection and transmission coefficients



Require continuity of tangential components of electric and magnetic intensity:

$$H - H' = H''$$

Express Hs in terms of Es:

$$kE - k'E' = k''E''$$

$$E\cos\theta + E'\cos\theta = E''\cos\phi$$

These two are equations for E' and E'' for given E. Solve ...

P-polarized case: reflection and transmission coefficients

$$r_\parallel \equiv r_p = \left[\frac{E'}{E}\right]_{TM} \qquad t_\parallel \equiv t_p = \left[\frac{E''}{E}\right]_{TM}$$

$$r_\parallel = -\frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t} \quad , \quad t_\parallel = \frac{2n_i \cos\Theta_i}{n_t \cos\Theta_i + n_i \cos\Theta_t}$$

Note: There are variations in the convention used to choose orientation of vector amplitudes. For example, in Hecht, $r_\parallel \rightarrow -r_\parallel$ due to opposite orientation of E'.

Normal incidence

$$r_\perp = \frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} \quad , \quad t_\perp = \frac{2n_i \cos\Theta_i}{n_i \cos\Theta_i + n_t \cos\Theta_t}$$

$$r_\parallel = -\frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t} \quad , \quad t_\parallel = \frac{2n_i \cos\Theta_i}{n_t \cos\Theta_i + n_i \cos\Theta_t}$$

reduces to:

$$r_\perp = \frac{n_i - n_t}{n_i + n_t} \quad , \quad t_\perp = \frac{2n_i}{n_i + n_t}$$

$$r_\parallel = \frac{n_i - n_t}{n_t + n_i} \quad , \quad t_\parallel = \frac{2n_i}{n_t + n_i}$$

in terms of relative index:

$$r_\perp = r_\parallel = \frac{1-n}{1+n} \qquad t_\perp = t_\parallel = \frac{2}{1+n}$$

Intensity reflectance an transmittance:

$$R = \frac{n_i E'^2}{n_i E^2} = \left(\frac{n-1}{n+1}\right)^2 = r^2 \qquad T = \frac{n_t E''^2}{n_i E^2} = \frac{4n}{(1+n)^2} = nt^2$$

Energy conservation:

$$R + T = 1$$

Forms of Fresnel Equations

$\vec{E}$ normal to the plane of incidence, TE or s-polarization:

$$r_\perp = \frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} \quad , \quad t_\perp = \frac{2n_i \cos\Theta_i}{n_i \cos\Theta_i + n_t \cos\Theta_t}$$

$\vec{E}$ in the plane of incidence, TM or p-polarization:

$$r_\parallel = -\frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t} \quad , \quad t_\parallel = \frac{2n_i \cos\Theta_i}{n_t \cos\Theta_i + n_i \cos\Theta_t}$$

$$r_\perp = -\frac{\sin(\Theta_i - \Theta_t)}{\sin(\Theta_i + \Theta_t)} \quad , \quad t_\perp = +\frac{2\sin\Theta_t \cos\Theta_i}{\sin(\Theta_i + \Theta_t)}$$

$$r_\parallel = -\frac{\tan(\Theta_i - \Theta_t)}{\tan(\Theta_i + \Theta_t)} \quad , \quad t_\parallel = +\frac{2\sin\Theta_t \cos\Theta_i}{\sin(\Theta_i + \Theta_t)\cos(\Theta_i - \Theta_t)}$$

$$r_s = \frac{\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_s = \frac{2\cos\Theta_i}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

$$r_p = -\frac{n^2\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_p = \frac{2n\cos\Theta_i}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

Reflectance and Transmittance

We have so far calculated field amplitudes. Now we answer the following:
Q: How much energy from an incident beam is reflected, and how much is transmitted?

Consider **reflection** first:

$$R = \frac{\frac{1}{2}\epsilon_0 n_1 c|E'|^2}{\frac{1}{2}\epsilon_0 n_1 c|E|^2} = \frac{|E'|^2}{|E|^2}$$

$$R_s = |r_s|^2 \qquad R_p = |r_p|^2$$

... and for unpolarized light:

$$\bar{R}(\theta_i) = \frac{1}{2}\left(R_S(\theta_i) + R_p(\theta_i)\right)$$

Next **transmission**:

This case a bit more complicated due to two facts:

- incident and reflected beams have different cross-sections
- they also propagate in different media

Look in the reference notes for the exact calculation. Here, we will *guess* the answer.

$$T = \left|\frac{E''}{E}\right|^2 \times \mathrm{CrossSectionFactor} \times \mathrm{MaterialFactor}$$

$$T = \left|\frac{E''}{E}\right|^2 \times \frac{\cos\theta_t}{\cos\theta_i} \times \frac{n_t}{n_i}$$

$$T_s = n|t_s|^2 \frac{\cos\theta_t}{\cos\theta_i} \qquad T_p = n|t_p|^2 \frac{\cos\theta_t}{\cos\theta_i}$$

Exercise:
Show that energy is conserved, and

$$R + T = 1$$

exactly as we should expect.