Helmholtz Equation

Helmholtz, Hermann Ludwig
1821 — 1894

- for situations with a well-defined color/wavelength/frequency
- eliminates (and thus simplifies) the temporal dimension of the problem
- is closely related to the wave equation

In complex representation:

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r},\omega)\exp[-i\omega t]$$

Here we "impose" harmonic dependence on time. Also called continuous wave (CW) regime.

Note:

- We use arguments of vector fields such as $\vec{E}(\vec{r},t)$ and $\vec{E}(\vec{r},\omega)$ to distinguish which quantity we are talking about.
- both are in general complex valued, and real physical fields are associated with their real parts.
- nontrivial time-dependent fields can be constructed as superpositions of components with different frequencies.

Helmholtz Equation

Let us see what the wave equation implies for $\vec{E}(\vec{r},\omega)$:

$$\left(\Delta - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = 0$$

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$$\left(\Delta - \frac{n^2}{c^2}(-i\omega)^2\right)\vec{E}(\vec{r},\omega)\exp[-i\omega t] = 0$$

and using the dispersion relation

$$k^2 = \frac{\omega^2 n^2(\omega)}{c^2}$$

leads to the Helmholtz equation:

$$\left[\Delta + k^2\right]\vec{E}(\vec{r},\omega) = 0$$

I many cases, it is an effective “replacement” for the (time-dependent) wave equation.

Paraxial approximation, and slowly varying envelope

Note: The following material answers our long-delayed question: *Which equations* are solved by Gaussian beams?

So, we have to solve

$$\left[\Delta + k^2\right] \vec{E}(\vec{r}, \omega) = 0 \ .$$

Assume we are looking at a laser beam propagating along z. A plane wave would behave as

$$\vec{E}(\vec{r}, \omega) \sim \exp[ikz] \qquad k = \frac{2\pi}{\lambda} \ ,$$

and we expect that this feature will be preserved also in more complex solutions. That is why we try the following ansatz:

$$\vec{E}(\vec{r}, \omega) = \vec{A}(x, y, z) \exp[ikz]$$

in which we expect that the dependence of $A(x, y, z)$ on z is "slow." This is called **slowly varying envelope**.

The resulting equation for A reads:

$$(\partial_{xx} + \partial_{yy}) A(x, y, z) + \partial_{zz} A(x, y, z) + 2ik\partial_z A(x, y, z) - k^2 A(x, y, z) + k^2 A(x, y, z) = 0$$

This is where we neglect $\partial_{zz} A(x, y, z)$ in comparison to $2ik\partial_z A(x, y, z)$ and thus obtain the **paraxial beam-propagation equation**:

$$\partial_z A(x, y, z) = \frac{i}{2k} (\partial_{xx} + \partial_{yy}) A(x, y, z)$$

Exercise:

Show that the Gaussian beam formula (derived before as a superposition of plane waves with small angles of propagation w.r.t. axis) is a solution to the paraxial beam propagation equation.

Hints:

- Make sure that you remove the carrier wave from the Gaussian beam formula
- This exercise is easier with the complex beam parameter representation
- Gaussian beam recap:

  In terms of complex beam parameter:

$$E(x,y,z,t) = E_0 e^{i\omega(z/c-t)} \frac{1}{q(z)} \exp\left[+ik\frac{x^2+y^2}{2q(z)}\right] \qquad q(z) = z - iz_R = z - i\frac{\pi w_0^2}{\lambda} \qquad k = \frac{\omega}{c}$$

  In terms of z-dependent beam size, and wavefront radius:

$$E(x,y,z,t) = E_0 e^{i\omega(z/c-t)} \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w(z)^2}\right] \exp\left[+i\frac{k(x^2+y^2)}{2R(z)}\right] e^{-i\arctan(z/z_R)}$$

  Important characteristics:

$$P = AI_0 \qquad A = \frac{\pi w_0^2}{2} \qquad z_R = \frac{\pi w_0^2}{\lambda} \qquad w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \qquad R(z) = z\left(1+\left(\frac{z_R}{z}\right)^2\right)$$

What next:

We will utilize Helmholtz to analyze continuous-wave regime problems, such as reflection from a material interface.