

## Recap: Our model of an “elementary dipole”

- Nucleus + Electron “cloud”
- Nucleus is heavy, and oblivious to the optical field
- Electron cloud experiences the Lorentz force  $\vec{F} = q_e \vec{E}$ , and responds to it in terms of an effective displacement  $\vec{x}$
- To account for the fact that electron is bound to the nucleus, we introduce a restoring force due to the Coulomb attraction:

$$F_C = -m_e \omega_0^2 \vec{x}$$

Note that this is nothing but a **harmonic oscillator**.  $m_e$  is electron mass, and the parameter  $\omega_0$  measures the strength of force.

- Refractive index as **a function of frequency**:

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2)}$$

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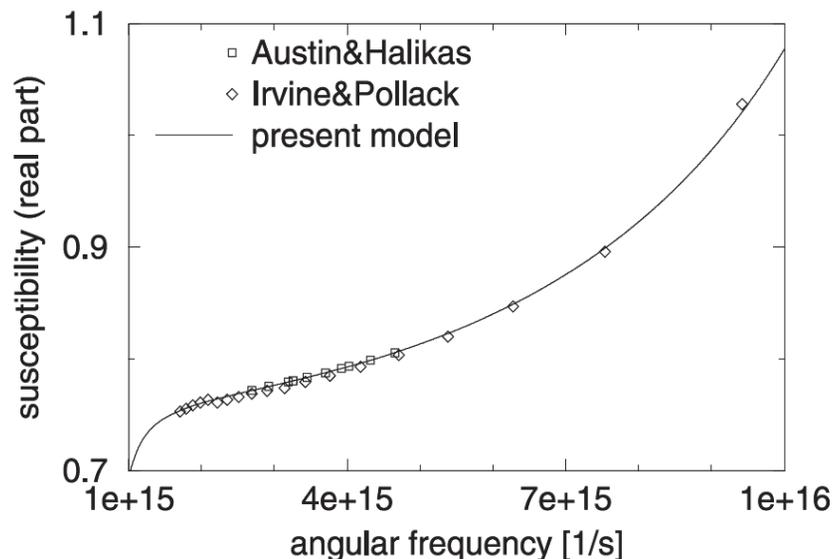
- Refractive index as **a function of frequency**:

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2)}$$

- **Problem:** We should account for energy dissipation in order to “heal” the singularity in the above refractive index expression
- **Question:** Since the dipoles (as an ensemble) can affect the propagating optical field, can they also affect each other?

## Refractive index in transparent media

- Practically speaking, (linear) refractive index as a function frequency can not be calculated accurately from first principles.
- In practice, experimental data are used to generate various formulas to represent  $n(\omega)$



- We want simple and *smooth* representations (often we need to take derivatives: e.g. the formula for the group velocity).
- Cauchy formula
- Sellmeier formula

**Cauchy formula:**

and assume that  $\omega_{0j} \gg \omega$  for all  $j$ . In other words, we are looking at

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2)},$$

specialized for *small*  $\omega$ . So, we can Taylor-expand around  $\omega = 0$ :

$$n(\omega) \approx A_0 + A_2\omega^2 + A_4\omega^4 + \dots$$

Use  $\omega = 2\pi c/\lambda$ :

$$n^2(\lambda) = C_0 + \frac{C_2}{\lambda^2} + \frac{C_4}{\lambda^4} + \dots$$

This is called **Cauchy formula** for the refractive index of a transparent dielectric.

**Example:** Fused silica has  $C_0 = 1.4580$  and  $C_2 = 0.00354$  (for  $\lambda$  in  $\mu\text{m}$ ).

**Note:** For practical reasons, we want these “fitting formulas” to have as few parameters as possible. Different functional forms may be suitable for different materials or wavelength regions...

**Exercise:** Calculate  $C_0$  and  $C_2$  from our “microscopic” formula for  $n(\omega)$ . Note that many transitions (many resonance terms) can contribute to each parameter.

## Sellmeier formula

Start from

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2)}$$

and use  $\omega = 2\pi c/\lambda$ :

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - (2\pi c/\lambda)^2)},$$

now multiply top and bottom of each term by

$$\frac{\lambda^2}{\omega_{0j}^2}$$

to get:

$$n^2(\omega) = 1 + \sum_j \frac{\frac{f_j N q_e^2}{\epsilon_0 m_e \omega_{0j}^2} \lambda^2}{(\lambda^2 - (2\pi c/\omega_{0j})^2)},$$

and collapse all unknowns into **Sellmeier coefficients**:

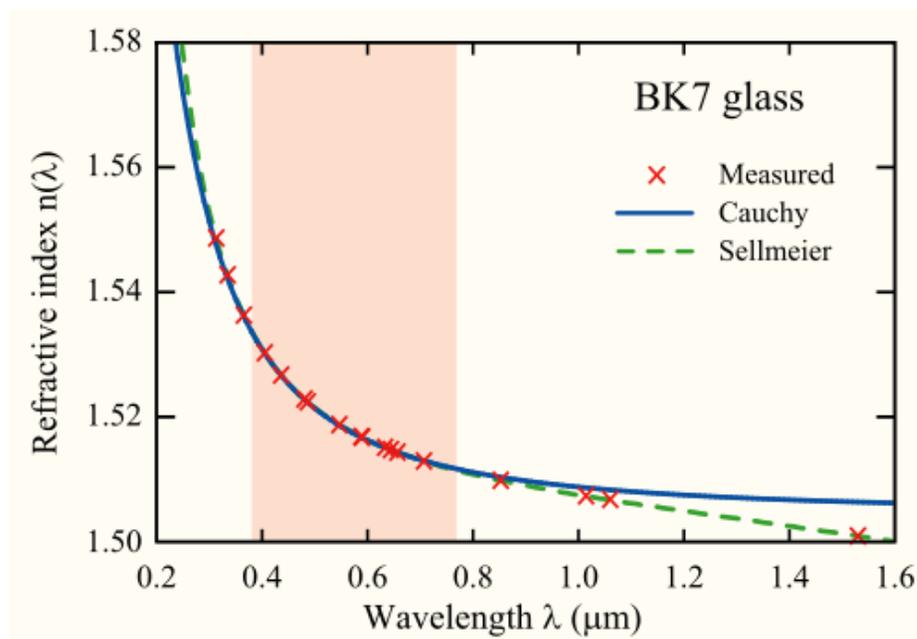
$$n^2(\omega) = 1 + \sum_j \frac{B_j \lambda^2}{(\lambda^2 - C_j)},$$

## Sellmeier formula

... is better (than Cauchy) because it better reflects the resonant structure of the refractive index.

### Note:

- Sellmeier coefficient values are tied to the wavelength unit used, usually  $\mu\text{m}$ .
- Any formula is only valid in a certain interval of wavelengths.
- The “nontrivial” part of the Sellmeier equation expresses susceptibility  $\chi$ .



## Accounting for losses

### Oscillator model with friction:

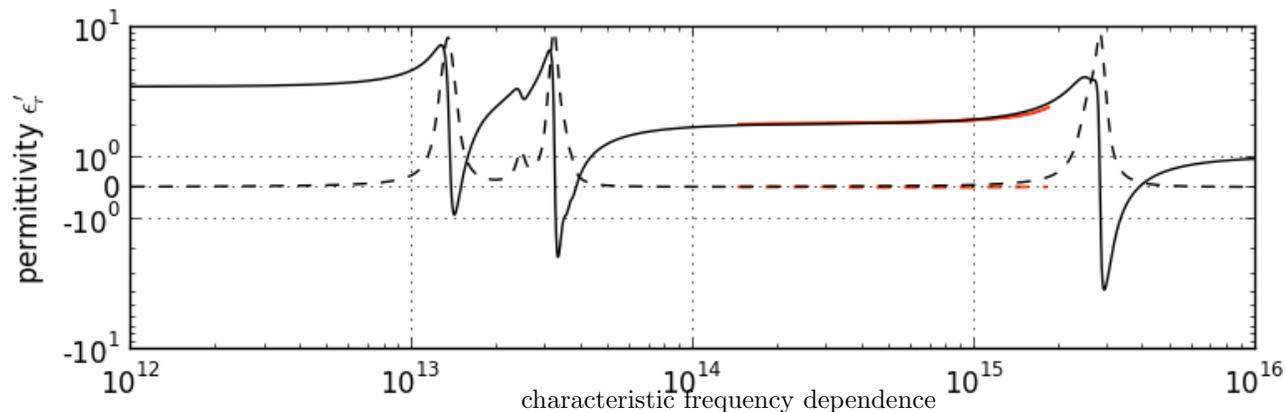
$$\vec{F} = m_e \vec{a} = m_e \frac{d^2 \vec{x}(t)}{dt^2} = q_e \vec{E}(t) - m_e \omega_0^2 \vec{x}(t) - \gamma m_e \frac{d\vec{x}(t)}{dt}$$

the last term causes velocity to decrease, and manifests itself as exponential damping of free oscillations. This provides a “channel” for the energy from the optical field to dissipate into the medium.

**Exercise:** Apply the same method we used previously in the friction-less case to show that the index of refraction becomes complex:

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2) - i\gamma\omega}$$

**Note:** The sign of the new term depends on the convention for the complex representation of optical fields.



## Consequences of complex refractive index

- nothing changes on the dispersion relation:

$$k = \frac{\omega n(\omega)}{c} = \frac{\omega[n_R(\omega) + in_I(\omega)]}{c}$$

- the wavevector therefore acquires imaginary part, too

$$k = k_R + ik_I$$

- ... and plane waves decrease in intensity with propagation distance:

$$E_0 \exp[ikx - i\omega t] = E_0 \exp[-k_I x] \exp[ik_R x - i\omega t]$$

- Intensity decays twice as fast;

$$I(x) = I(0) \exp[-2k_I x]$$

- Intensity can also **increase**, if the imaginary part of the index becomes negative — this occurs **in gain media**, e.g. in semiconductor lasers.

## Mutual interaction between induced dipoles

- Induced dipole: reaction to the local electric field.
- local field consists of the imposed external field, plus contributions from all other dipole-fields.

The result of “summation” is the **Clausius-Mossotti formula**:

$$\frac{n^2(\omega) - 1}{n^2(\omega) + 2} = \frac{Nq_e^2}{3\epsilon_0 m_e} \sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2 + i\gamma\omega)}$$

**Note:** If the medium is sufficiently dilute, dipoles are far from each other, and should not influence the local field much. Also, the refractive index will be close to one. In such a case the Clausius-Mossotti formula reduces to the expression we derived earlier.



Clausius, Rudolf  
1822 — 1888



Mossotti, Ottaviano-Fabrizio  
1791 — 1863

## Birefringence in crystals

- So far we have assumed that our medium model was **isotropic**, i.e. all directions were equivalent (or indistinguishable).
- For atoms in crystals, their immediate neighborhood is anisotropic, and that means that our  $\omega_0$  can depend on the direction of oscillation.
- This means that we can have different

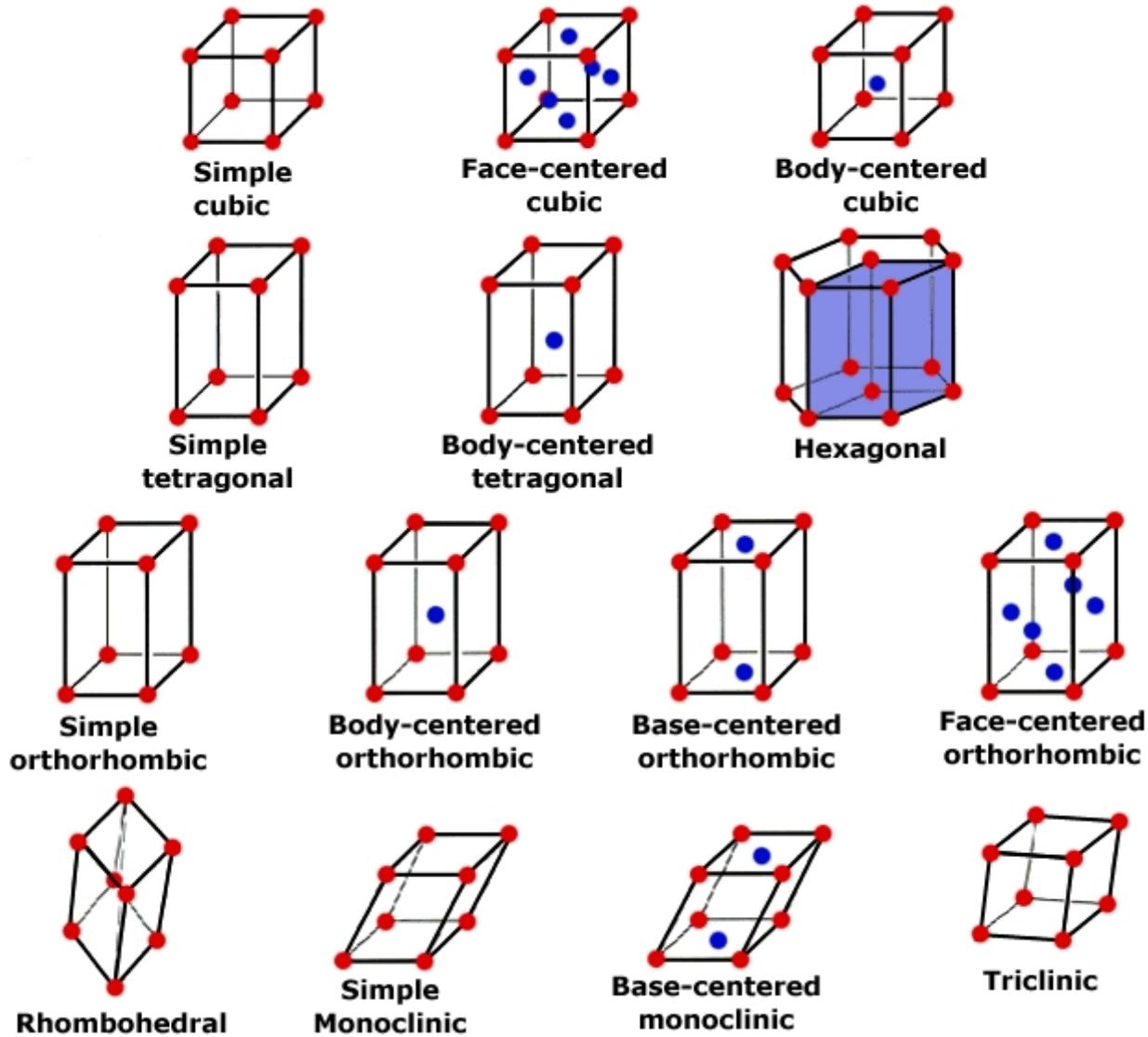
$$\omega_{0x} \quad \omega_{0y} \quad \omega_{0z}$$

- ... and that we have different refractive indices  $n_x$ ,  $n_y$ , and  $n_z$ , e.g.

$$n_x^2 \approx 1 + \frac{Nq_e^2}{\epsilon_0 m_e \omega_{0x}^2}$$

- This is called **birefringence**, and it occurs in anisotropic crystals where the atoms are arranged in unit cells that have low symmetry.
- **Note:** refractive index depends on the **direction of oscillation**. That means that in general, there can be **different refractive index** for the same direction of propagation.

## Crystal units cells - Bravais lattices



**Cubic crystals:**

$$n_x = n_y = n_z$$

*Macroscopic* properties in these media are **isotropic**.

**Uniaxial crystals:**

$$n_o = n_x = n_y \neq n_z = n_e$$

measure of birefringence:

$$\Delta n = (n_e - n_o)$$

depending on its sign, we distinguish **positive** and **negative** uniaxial crystals

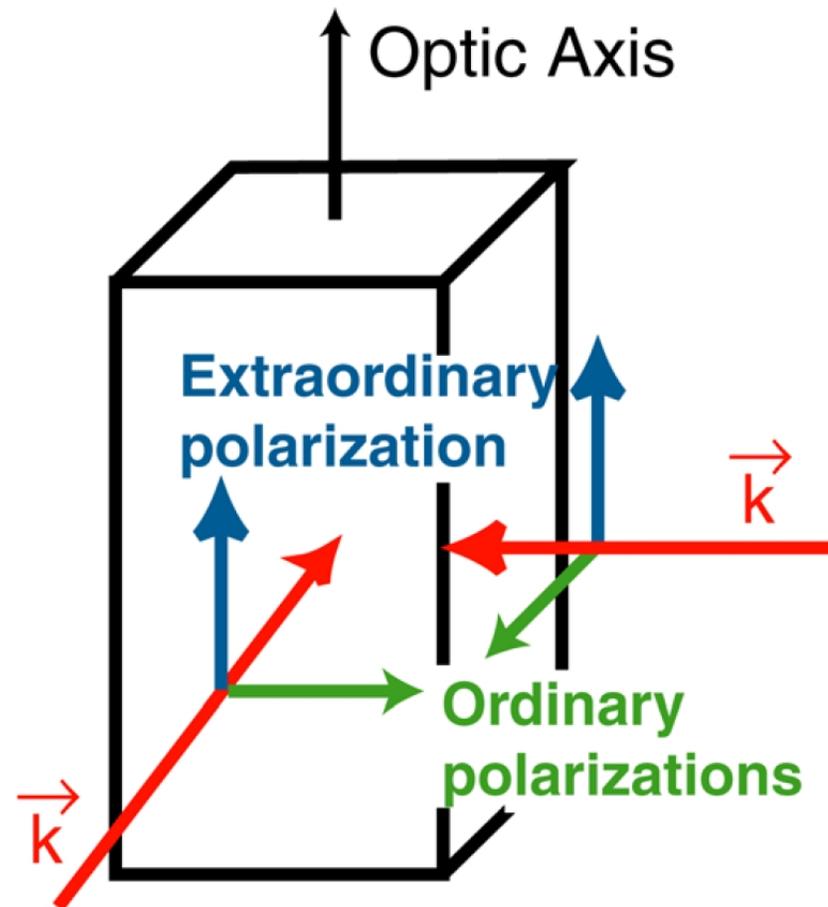
**Biaxial crystals:**

$$n_x \neq n_y \neq n_z \neq n_x$$

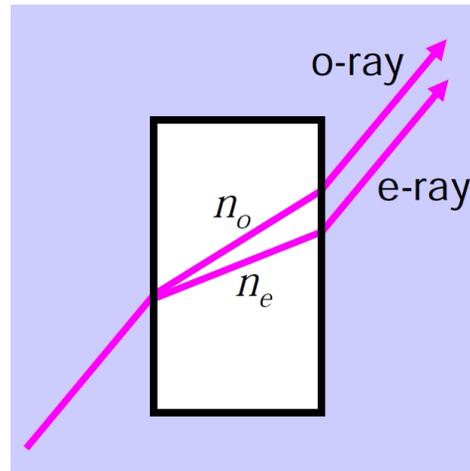
## Uniaxial crystals:

- Have one refractive index that is different (from the other two).
- Its direction is the direction of **optical axis**.
- Light polarized along the optical axis is called **extraordinary ray**, and experiences the refractive index  $n_e$ .
- Light that is polarized perpendicularly to the axis is called **ordinary ray**, and experience refractive index  $n_o$ .
- These polarization directions are the **crystal principal axes**
- Light with any other polarization must be broken down into its ordinary and extraordinary components.
- We perform calculations with each of them individually, using appropriate refractive index, and recombine them afterward.

## Uniaxial crystals, polarization decomposition:



Birefringence can separate the two polarizations into separate beams



**Uniaxial crystals, examples:**

Crystal	$n_o$	$n_e$
Calcite ( $\text{CaCO}_3$ )	1.658	1.486
Lithium niobate ( $\text{LiNbO}_3$ )	2.286	2.200
Lithium tantalite ( $\text{LiTaO}_3$ )	2.176	2.180
Quartz ( $\text{SiO}_2$ )	1.544	1.553
Rutile ( $\text{TiO}_2$ )	2.616	2.903
Beryl ( $\text{Be}_3\text{Al}_2$ )	1.602	1.557
Calomel ( $\text{Hg}_2\text{Cl}_2$ )	1.973	2.656
Magnesium Fluoride ( $\text{MgF}_2$ )	1.380	1.385
Sapphire ( $\text{Al}_2\text{O}_3$ )	1.768	1.760
Sodium Nitrate ( $\text{NaNO}_3$ )	1.587	1.336
Peridot ( $\text{Mg, Fe}$ ) $\text{SiO}_4$	1.690	1.654
Ice ( $\text{H}_2\text{O}$ )	1.309	1.313

**Interesting consequences of anisotropy:**

- Electric field  $\mathbf{E}$  and electric induction  $\mathbf{D}$  are not parallel (in general)
- Gauss dictates that wave-vector  $\vec{k}$  is perpendicular to  $\mathbf{D}$ , which in turn means that  $\mathbf{E}$  and  $\vec{k}$  need not be perpendicular ...
- ... and Poynting  $\vec{S}$  will have direction different from that of  $\vec{k}$ .

... we will justify and clarify these observations in solving simple problem examples ...

## Examples: Wave Propagation in a Uniaxial Medium

**A** propagation along optic axis:

all wave-polarizations “see” the same refractive index

**B** propagation direction perpendicular to optic axis:

orthogonal polarizations experience different refractive indices

**C** propagation at angle w.r.t. optic axis:

ordinary wave polarized perpendicular to the axis “sees”  $n_o$

extraordinary wave experiences refractive index that depends on the propagation direction

## Problem Example:

This is to explore regime **B** in some detail. Consider a slab of uni-axial crystal, thickness  $L$ , with the optic axis parallel to its surface, and refractive indices  $n_o$  and  $n_e$ . An electromagnetic plane wave is incident from vacuum, in direction normal to the surface of the slab. Given are:

the electric field amplitude  $E_0$

wavelength  $\lambda$

linear polarization direction 45 deg w.r.t. optic axis

electric field value is zero at  $t = 0$  at the surface of the crystal

Also assume that no reflections occur at material interfaces.

- A) Write the complex representation of the incoming plane wave solution
- B) Find the plane wave solution inside the crystal. You should observe that the polarization of the wave depends on the propagation distance.
- C) Find the condition that the thickness  $L$  has to satisfy in order to have the outgoing plane wave polarized at 90 deg w.r.t. the incoming wave polarization.