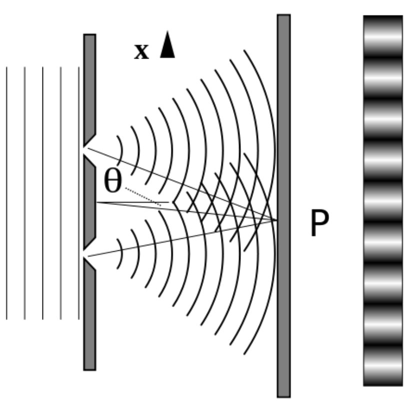
Regimes in optical interference

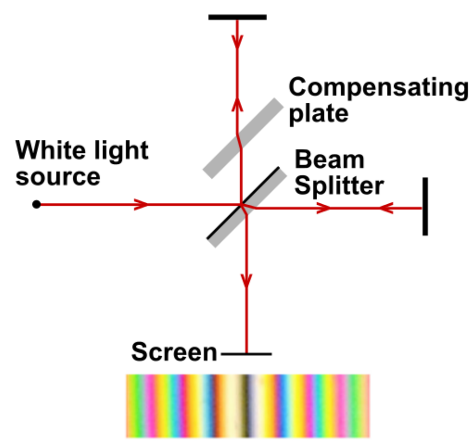
From our examination of interference between solutions of the 3D wave equation:

- frequencies must be very similar. In practice it means that in the optical regime the two waves must be "derived" from the same source.
- interference patterns attain their deepest modulation when the two interfering waves have comparable amplitudes

Ways to obtain waves ready for interference:

- Division of wavefront: Separate "pieces" from the same-wave wavefront and bring them to overlap and interfere at a different location
- Division of amplitude: Leave the wavefront intact, but split it into two (or more) waves with smaller amplitudes and let them propagate along different paths. The splitting occurs at the same point in space.





Interference between two optical fields

Electric field of a two-wave superposition:

$$\vec{E}(\vec{r},t) = \hat{e}_1 E_1 \cos[\vec{k}_1.\vec{r} - \omega_1 t + p_1] + \hat{e}_2 E_2 \cos[\vec{k}_2.\vec{r} - \omega_2 t + p_2]$$

Detecting the magnitude of Poynting vector:

$$\frac{S(\vec{r},t)}{\epsilon_0 n c} = \vec{E}(\vec{r},t).\vec{E}(\vec{r},t)$$

Identify three contributions:

$$S = S_1 + S_2 + S_{12}$$

$$S \sim E_1^2 \cos^2[\vec{k}_1.\vec{r} - \omega_1 t + p_1] + E_2^2 \cos^2[\vec{k}_2.\vec{r} - \omega_2 t + p_2] + 2(\hat{e}_1.\hat{e}_2) E_1 E_2 \cos[\vec{k}_1.\vec{r} - \omega_1 t + p_1] \cos[\vec{k}_2.\vec{r} - \omega_2 t + p_2]$$

Obviously, S_1 and S_2 are magnitudes of Poynting of individual waves. S_{12} is the **interference term**.

Fresnel-Arago law:

If the two fields have orthogonal polarization $(\hat{e}_1.\hat{e}_2) = 0$, there is no interference.

Next, take:

$$\hat{e}_1 = \hat{e}_2 \qquad p_1 = p_2 = 0$$

$$S \sim E_1^2 \cos^2[\vec{k}_1.\vec{r} - \omega_1 t] + E_2^2 \cos^2[\vec{k}_2.\vec{r} - \omega_2 t] + 2E_1E_2 \cos[\vec{k}_1.\vec{r} - \omega_1 t] \cos[\vec{k}_2.\vec{r} - \omega_2 t]$$

Now use:

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

so that the interference term becomes:

$$S_{12} \sim \cos[\vec{k}_1.\vec{r} - \omega_1 t] \cos[\vec{k}_2.\vec{r} - \omega_2 t] = \frac{1}{2}\left[\cos[(\vec{k}_1 + \vec{k}_2).\vec{r} - (\omega_1 + \omega_2)t] + \cos[(\vec{k}_1 - \vec{k}_2).\vec{r} - (\omega_1 - \omega_2)t]\right]$$

The first term is fast oscillating and the second is slow oscillating as long as $\omega_1 \sim \omega_2$.

Time averaging: First term lost. S_1 and S_2 attain their independent time-averaged values:

$$\frac{< S_1(\vec{r}, t) >_T}{\epsilon_0 nc} = \frac{1}{2}E_1^2 \qquad \frac{< S_2(\vec{r}, t) >_T}{\epsilon_0 nc} = \frac{1}{2}E_2^2$$

and

$$\frac{< S_{12}(\vec{r}, t) >_T}{\epsilon_0 nc} = E_1E_2 \cos[\Delta\vec{k}.\vec{r} - \Delta\omega t] \qquad \Delta\vec{k} = (\vec{k}_1 - \vec{k}_2) \qquad \Delta\omega = (\omega_1 - \omega_2)$$

Q: For the above to be OK, we had to assume something about the length of the averaging time. What?

So the final result is:

$$< S >_T = I_1 + I_2 + 2\sqrt{I_1 I_2}(\hat{e}_1.\hat{e}_2)\cos[\Delta\vec{k}.\vec{r} - \Delta\omega t]$$

with

$$\Delta\vec{k} = (\vec{k}_1 - \vec{k}_2) \qquad \Delta\omega = (\omega_1 - \omega_2)$$

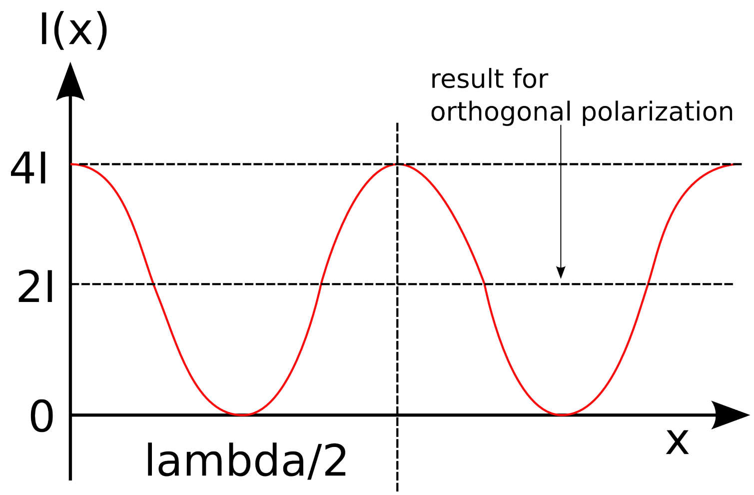
Example, Standing waves:

Take

$$\vec{k}_1 = +k\hat{i} \quad , \quad \vec{k}_2 = -k\hat{i} \quad , \quad \omega_1 = \omega_2 \quad , \quad I_1 = I_2 = I$$

so that we have two identical waves (counter-) propagating in opposite directions.

$$< S >_T = 2I(1 + \cos[2kx - 0t])$$



Problem example:

A) Generalize the above derivation of the interference pattern for $I_1 \neq I_2$ and sketch the intensity profile as a function of the coordinate x, making sure to mark all important scales such as minimal and maximal intensity, intensity modulation depth, and the spatial period of the fringe pattern.

B) Generalize the above derivation for $\omega_1 \neq \omega_2$. Show that the resulting interference pattern will be moving. Determine this velocity, and estimate its value for two *close* wavelength from the visible region.

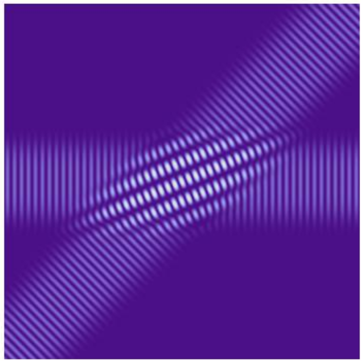
C) Which direction is the pattern moving? With, or against the direction of the wave with the higher frequency?

Example, nearly co-propagating waves

Now we consider two waves (same frequency ω) which propagate in the same general direction. Of course, the formula remains valid...
What is the spacing L between the fringes in this case?

$$|\vec{k}_1 - \vec{k}_2|L = \Delta k L = 2\pi$$

$$L = \frac{2\pi}{\Delta k}$$



Note: In this case, the fringes can be "arbitrarily" thick, the smaller the angle between the waves, the larger L gets.

Problem example: For the case of nearly co-propagating waves with very close frequencies, derive the expression for the velocity of fringes (in the direction perpendicular to the common "mean" direction of propagation) as a function of the mean angular frequency and the angle between the waves.

Hint: Take advantage of the fact that the angle is small, as is the frequency difference.

Co-propagating waves:

Note: This is a simplified view of a wavepacket: In general, they are realized as a continuos (integral) superposition of infinitely many waves. In the present case, we are dealing with "discrete, or finite" superposition.

This time we will look at the *field itself* for the case

$$\vec{k}_1 = k_1\hat{i} \quad \vec{k}_2 = k_2\hat{i} \quad p_2 = p_2 = 0 \quad \hat{e}_2 = \hat{e}_2 \quad \vec{E}_1 = E_2 = E_0$$

$$\vec{E}(\vec{r},t) = \hat{e}E_0\left[\cos(k_1x - \omega_1 t) + \cos(k_2x - \omega_2 t)\right]$$

Note: Of course, we can use the formula for interference of two scalar waves derived previously... ($A = B$, and $\hat{e}$ does not really matter)

$$\vec{E}(\vec{r},t) = 2\hat{e}E_0\cos\left[\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right]\cos\left[\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right]$$

Define

$$\bar{k} = \frac{k_1+k_2}{2} \qquad \bar{\omega} = \frac{\omega_1+\omega_2}{2}$$

and

$$\Delta k = \frac{k_1-k_2}{2} \qquad \Delta\omega = \frac{\omega_1-\omega_2}{2}\ .$$

... and obtain what should be now a familiar expression in terms of **carrier** and **envelope**:

$$\vec{E}(\vec{r},t) = 2\hat{e}E_0\cos(\bar{k}x - \bar{\omega}t)\cos(\Delta kx - \Delta\omega t)$$

Carrier "motion":

$$\cos(\bar{k}x - \bar{\omega}t) \rightarrow \cos[\bar{k}(x - \frac{\bar{\omega}}{\bar{k}}t)]$$

this gives us (similar frequencies!) the **phase velocity**:

$$v_p = \frac{\bar{\omega}}{\bar{k}} = \frac{\omega}{k}$$

Envelope "motion":

$$\cos(\Delta kx - \Delta\omega t) \rightarrow \cos[\Delta k(x - \frac{\Delta\omega}{\Delta k}t)]$$

this gives us the **group velocity**:

$$v_g = \frac{\Delta\omega}{\Delta k} \rightarrow \frac{\partial\omega}{\partial k}$$

Note: In relations as the one above, ω and k are viewed as functions of each other, **defined by the dispersion relation:**

$$k = \frac{\omega(k)n(\omega(k))}{c} \qquad \text{or} \qquad k(\omega) = \frac{\omega n(\omega)}{c}$$

The independent variable is k on the left, and ω on the right.
Note: The second relation is easier to use (**Q:** Why?). That is why we usually calculate the inverse of the group velocity as

$$\frac{1}{v_g} = \frac{\partial k}{\partial\omega}$$

Relation between group and phase velocities

Start from

$$\frac{1}{v_g} = \frac{\partial k}{\partial \omega}$$

and execute the derivative,

$$\frac{1}{v_g} = \frac{\partial k}{\partial \omega} = \frac{n(\omega)}{c} + \frac{\omega}{c}\frac{\partial n(\omega)}{\partial \omega} = \frac{n(\omega)}{c} + \frac{n(\omega)}{c}\frac{\omega}{n(\omega)}\frac{\partial n(\omega)}{\partial \omega} = \frac{1}{v_p}\left(1 + \frac{\omega}{n(\omega)}\frac{\partial n(\omega)}{\partial \omega}\right)$$

so the two velocities are related by

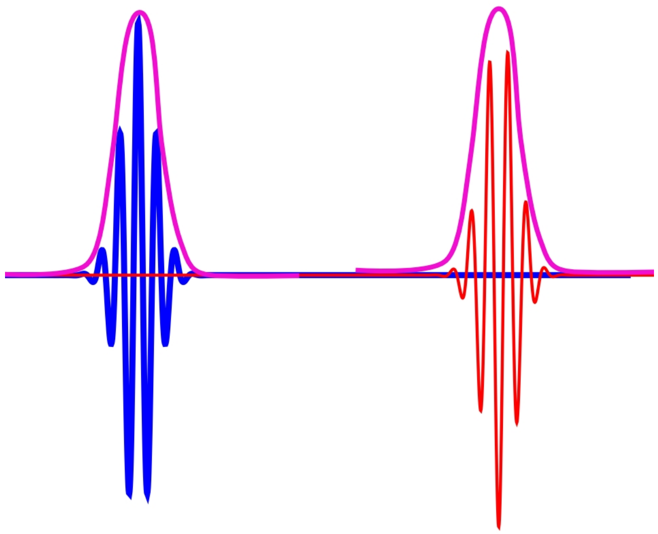
$$v_g = v_p \frac{1}{\left(1 + \frac{\omega}{n(\omega)}\frac{\partial n(\omega)}{\partial \omega}\right)}$$

Observation: Refractive index as a function of the angular frequency determines not only the dispersion relation, but also group and phase velocity. The two velocities can be quite different.

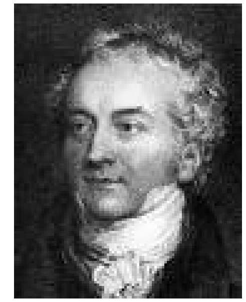
Note: Recall the example of wave-guided 3D wave equation solution: It also showed that group and phase velocities were different. In that case the reason was the geometry of the waveguide. In general **material** and **waveguide dispersion** both contribute to propagation properties of EM waves.

These issues will be revisited in detail in the Section on light-matter interaction.

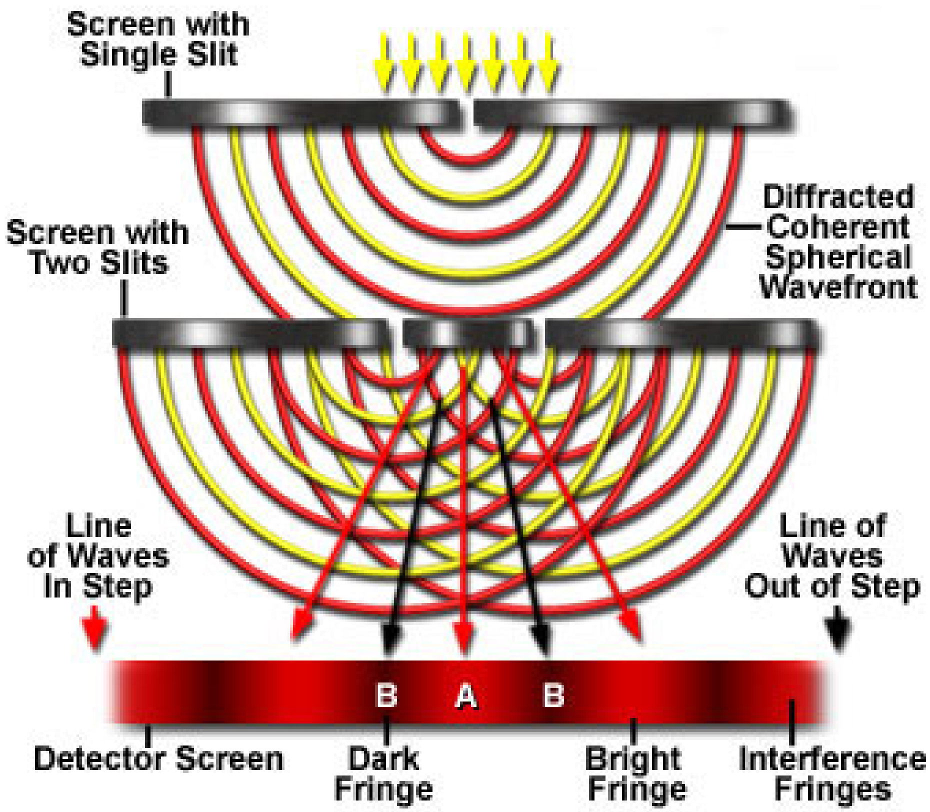
Relation between group and phase velocities



- Wave-packet as a whole moves with the group velocity.
- The carrier oscillation "underneath" the envelope runs with the phase velocity
- When $v_p \neq v_g$ the relative position of the carrier and envelope (called **Carrier-Envelope Phase (CEP)**) changes upon propagation.
- This is best visible in short-duration (few-cycle) pulses.
- Because $n(\omega)$ depends on frequency in non-trivial way (which we will explore in Section 6), also $v_p(\omega)$ and $v_g(\omega)$ depend on frequency or wavelength.
- This dependence in v_g is called **group velocity dispersion**. It causes pulses of different color to take different time to travel the same distance (e.g. in fibers).


Young, Thomas
1773 — 1829

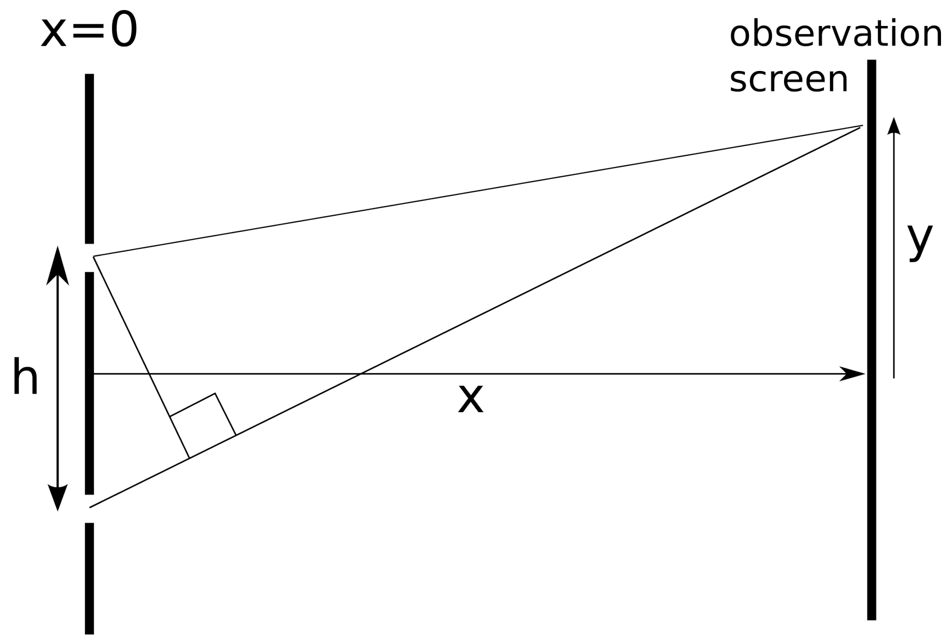
Young‘s two-slit experiment



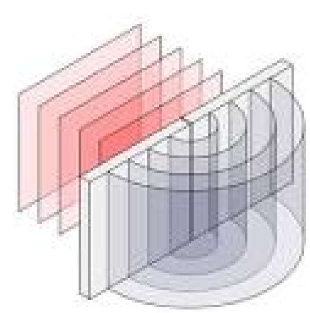
- This is an example **division of wavefront**.
- Displays wave nature of EM field

Far-field pattern examination

geometry, notation:



- source of light is distant
- field polarized along z
- observation screen is **far**, $x >> h$, **and** $x >> y$
- slits are very narrow, they act as sources of cylindrical waves:



At the observation point, two waves originating in the two slits add:

$$\vec{E}(\vec{r},t) = \frac{A\hat{k}}{|\vec{r}-\hat{j}h/2|}\exp i[k|\vec{r}-\hat{j}h/2| - \omega t + p] + \frac{A\hat{k}}{|\vec{r}+\hat{j}h/2|}\exp i[k|\vec{r}+\hat{j}h/2| - \omega t + p]$$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$|\vec{r} \pm \hat{j}h/2| = \sqrt{x^2 + (y \pm h/2)^2}$$

Now we use the fact that $x >> y, h$, and Taylor expand:

$$|\vec{r} \pm \hat{j}h/2| = \sqrt{x^2 + (y \pm h/2)^2} \approx x\left(1 + \frac{(y \pm h/2)^2}{2x^2}\right)$$

here we used a fromula worth to remember, $\sqrt{a^2+b^2} = a\sqrt{1+b^2/a^2} \approx a(1+b^2/(2a^2)) = a + b^2/(2a)$

$$|\vec{r} \pm \hat{j}h/2| \approx x\left(1 + \frac{y^2}{2x^2} \pm \frac{yh}{2x^2} + \frac{h^2}{8x^2} + \dots\right)$$

$$|\vec{r} \pm \hat{j}h/2| \approx L \pm \frac{yh}{2x^2} \qquad L = x + \frac{y^2}{2x} + \frac{h^2}{8x}$$

Note: As a general rule, we have to be more precise in the phase arguments of exponentials (or sines/cosines), while a rougher approximations are sufficient in the amplitude pre-factors.

$$\vec{E}(\vec{r},t) = \hat{k}\frac{A}{x}\exp[i(kL - kyh/2x - \omega t + p)] + \hat{k}\frac{A}{x}\exp[i(kL + kyh/2x - \omega t + p)]$$

$$\vec{E}(\vec{r},t) = \hat{k}\frac{2A}{x}\exp[i(kL - \omega t + p)]\cos\left[\frac{kyh}{2x}\right]$$

The observable is the time-averaged Poynting. To simplify calculation, note that we are not going to get an absolutely correct result — so why not to throw away unimportant (for now) common pre-factors. We look at the *profile* of the intensity

$$< S >_T \sim < \vec{E}.\vec{E}^* > = \sim \left(\frac{2A}{x}\right)^2 \frac{1}{2}\cos^2\left[\frac{kyh}{2x}\right]$$

The argument of cos is

$$\frac{kyh}{2x} = \frac{2\pi}{\lambda}\frac{yh}{2x} = \pi\frac{yh}{\lambda x}$$

Determine positions of **bright fringes**:

$$\pi\frac{yh}{\lambda x} = m\pi \qquad m = 0, \pm 1, \pm 2, \dots$$

$$y = m\frac{\lambda x}{h} \qquad \Delta y = \frac{\lambda x}{h}$$

Lesson: Adding harmonic waves leads to optical interference, with fringes which depend on wavelength and geometry of the problem. These properties are at the heart of much optics — interferometers, metrology, ...

Note:

- fringe separation increases with wavelength
- it decreases with the characteristic scale (dimension) of the source, namely h
- this behavior is generic