Problem: This example deals with material that is an extension of OPTI-310. We have described the main properties of Gaussian beams, and noted that they are approximate solutions to wave and Maxwell equations. In the following, we go through their derivation, and identify exactly what approximations underline these beams.

Method: Construct Gaussian solution as approximate plane-wave superpositions. We will draw on the experience with the example of one-dimensional Gaussian wave-packet from the Section of 1D wave equation solutions.

Approximation: Laser beams exhibit almost perfect directionality. That means that the rays that constitute them propagate at very low angles w.r.t. their axis. This is called **paraxial propagation** and we will assume that the solution we are looking for is of that kind.

Solution:

Form a plane wave superposition

$$E = E_0 \iint dk_x dk_y A(k_x, k_y) \exp i[k_x x + k_y y + k_z z - \omega t]$$

with $A(k_x, k_y)$ (called spatial spectrum) to be "guessed" to obtain Gaussian solutions.

Note: This is already a **scalar approximation:** we only consider one component of the field, e.g. its x-component, and our beam will be linearly polarized.

Use dispersion relation

$$k^2 = \frac{\omega^2}{c^2} \to k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

Paraxial approximation: All rays are assumed to propagate with a shallow angle w.r.t. axis. It means that $k_x, k_y << k_z \approx \omega/c$. So we can Taylor expand and approximate k_z as:

$$k_z = \frac{\omega}{c} - \frac{1}{2}\frac{c(k_x^2 + k_y^2)}{\omega}$$

Use this in the superposition integral:

$$E \sim \iint dk_x dk_y A(k_x, k_y) \exp i[k_x x + k_y y + (\frac{\omega}{c} - \frac{1}{2}\frac{c(k_x^2 + k_y^2)}{\omega})z - \omega t]$$

Pull out all constants (i.e. terms independent of integration variables k_x, k_y), and collect like terms:

$$E \sim e^{i\omega(z/c-t)} \iint dk_x dk_y A(k_x, k_y) \exp i[k_x x + k_y y - \frac{1}{2}\frac{c(k_x^2 + k_y^2)}{\omega} z]$$

Use the rule of 90% of theoretical physics: Convert to a Gaussian integral. Apply a Gaussian shaped ansatz (recall our 1D wave-packet example for motivation):

$$A(k_x, k_y) = \exp[-\alpha k_x^2] \exp[-\alpha k_y^2]$$

The question is if we can find α such that we obtain Gaussian solutions?
With the ansatz, the above superposition can be written in a convenient form:

$$E \sim \int dk_x \exp[-k_x^2(\alpha + \frac{icz}{2\omega}) + ixk_x] \times \int dk_y \exp[-k_y^2(\alpha + \frac{icz}{2\omega}) + iyk_y]$$

Recall the Gaussian integral formula we used in the example with a 1D wave-packet,

$$\int dx \exp[-x^2 a + bx] = \sqrt{\frac{\pi}{a}} \exp[b^2/4a]$$

... and get:

$$E \sim \frac{\pi}{\alpha + \frac{icz}{2\omega}} \exp[-\frac{x^2 + y^2}{4(\alpha + \frac{icz}{2\omega})}]$$

Now we can fix α: For $z = 0$ we require that

$$E(z = 0) \sim \exp[-(x^2 + y^2)/w_0^2]$$

and this we get by choosing $4\alpha = w_0^2$.

Written in a neat form, the Gaussian (approximate) solution to wave equation reads

$$E \sim e^{i\omega(z/c-t)} \frac{1}{q(z)} \exp[-ik\frac{x^2+y^2}{2q(z)}] \qquad q(z) = z - iz_R = z - i\frac{\pi w_0^2}{\lambda} \qquad k = \frac{\omega}{c}$$

Note: $q(z)$ is called **complex beam parameter**. From here we will easily obtain other useful representations of Gaussian beams.

Exercise: Starting from the Gaussian beam solution expressed in term of a complex beam parameter, derive the following more explicit form:

$$E(x,y,z,t) = E_0 e^{i\omega(z/c-t)} \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w(z)^2}\right] \exp\left[+i\frac{k(x^2+y^2)}{2R(z)}\right] e^{-i\phi(z)}$$

where the beam size $w(z)$ and radius $R(z)$ of the wavefront are as specified in the Gaussian beam summary, and

$$\phi(z) = \arctan(z/z_R)$$

is **Gouy phase**.

Hint: This is all straightforward algebra, but Gouy phase might be puzzling. In your calculations, it will originate from the pre-factor containing the complex beam parameter.

Note: We have treated the Gaussian beam as a scalar field. We usually assign to it linear polarization vector as in

$$\vec{E}(x,y,z,t) = \hat{e} E_0 \ldots$$

Note: Gaussian beam formulas depend on the notation used for the carrier wave, in our case $e^{i\omega(z/c-t)}$. If the opposite phase is used, it will change signs of all phases...

Exercise: We have used two approximations in the derivation of the Gaussian beam formula. One of them is that the solution is linearly polarized everywhere in space.

A) provide a qualitative argument why this can not be the case if the solution satisfied all Maxwell equations exactly.

B) identify which of Maxwell equations is violated and show this by explicit calculation.