Based on our experience with plane waves, we can construct vectorial plane wave solutions to the WE for **E** and **B**:

$$\begin{aligned} \mathbf{E}(\vec{r},t) &= \vec{E}_0 \exp i[\vec{k}.\vec{r} - \omega t] \\ \mathbf{B}(\vec{r},t) &= \vec{B}_0 \exp i[\vec{k}.\vec{r} - \omega t] \ . \end{aligned}$$

Having seen such WE solutions, we already know that the dispersion relation must be fulfilled:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \ .$$

Now, what about $\vec{E}_0$, $\vec{B}_0$? To find out constraints on these, we must *go back into ME*:

$$\begin{aligned} \nabla \times \mathbf{B} &= +\frac{1}{c^2}\partial_t \mathbf{E} \qquad \frac{1}{c^2} \equiv \epsilon_0 \mu_0 \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

... and use the operator equivalencies

$$\nabla \times \to i\vec{k} \times \qquad \nabla \cdot \to i\vec{k}. \qquad \partial_t \to -i\omega$$

for faster calculation to get ...

... plane-wave relation for vector amplitudes:

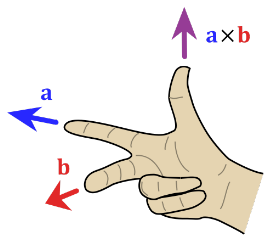
$$\vec{k} \times \vec{B}_0 = -\tfrac{1}{c^2}\omega \vec{E}_0 \quad \text{or} \quad \vec{E}_0 = -\frac{c^2}{\omega}\vec{k} \times \vec{B}_0$$
$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \quad \text{or} \quad \vec{B}_0 = +\frac{1}{\omega}\vec{k} \times \vec{E}_0$$

$$\vec{k}.\vec{E}_0 = 0$$
$$\vec{k}.\vec{B}_0 = 0$$

Q: If the first two equations are satisfied, the second pair is, too. It seems we did not really need the divergence constraints! How come?
A: Because divergence equations are merely constraints on initial conditions.

Note:

- It follows than in a plane wave, $c|\vec{B}_0| = |\vec{E}_0|$.
- Relative spatial orientation of $\vec{k}$, $\vec{E}_0$, $\vec{B}_0$ (in this order!) is the same as between $\hat{i}$, $\hat{j}$, $\hat{k}$
- These constitute a right-hand oriented system



Transverse wave summary:

Nomenclature:

$$\begin{aligned}\vec{E}(\vec{r},t) &= \hat{e}A \quad \exp[i(\vec{k}.\vec{r}-\omega t+p)]\\ \vec{B}(\vec{r},t) &= \hat{b}A/c\exp[i(\vec{k}.\vec{r}-\omega t+p)]\end{aligned}$$

- $\hat{e}$ unit vector in direction of E-field. Note that later in this course it may be a complex-valued vector.
- $\vec{k}$ propagation vector (sometimes wave-vector). It gives the direction of propagation, and its magnitude specifies the wave‘s spatial frequency along that direction.
- ω angular (temporal) frequency
- A amplitude (this, too, can be complex-valued!)
- p phase

As always: Real part has the meaning of the real physical field:

$$\vec{E}(\vec{r},t) = \hat{e}A\cos[\vec{k}.\vec{r}-\omega t+p]$$

Q: In the above line, I have silently assumed something. What is it?

Transverse wave summary cont.:

Vector amplitude properties:

- EM plane-waves are transverse:

$$\vec{k}.\hat{e}A\exp[i(\vec{k}.\vec{r}-\omega t+p)]=0 \qquad \vec{k}.\hat{e}=0$$

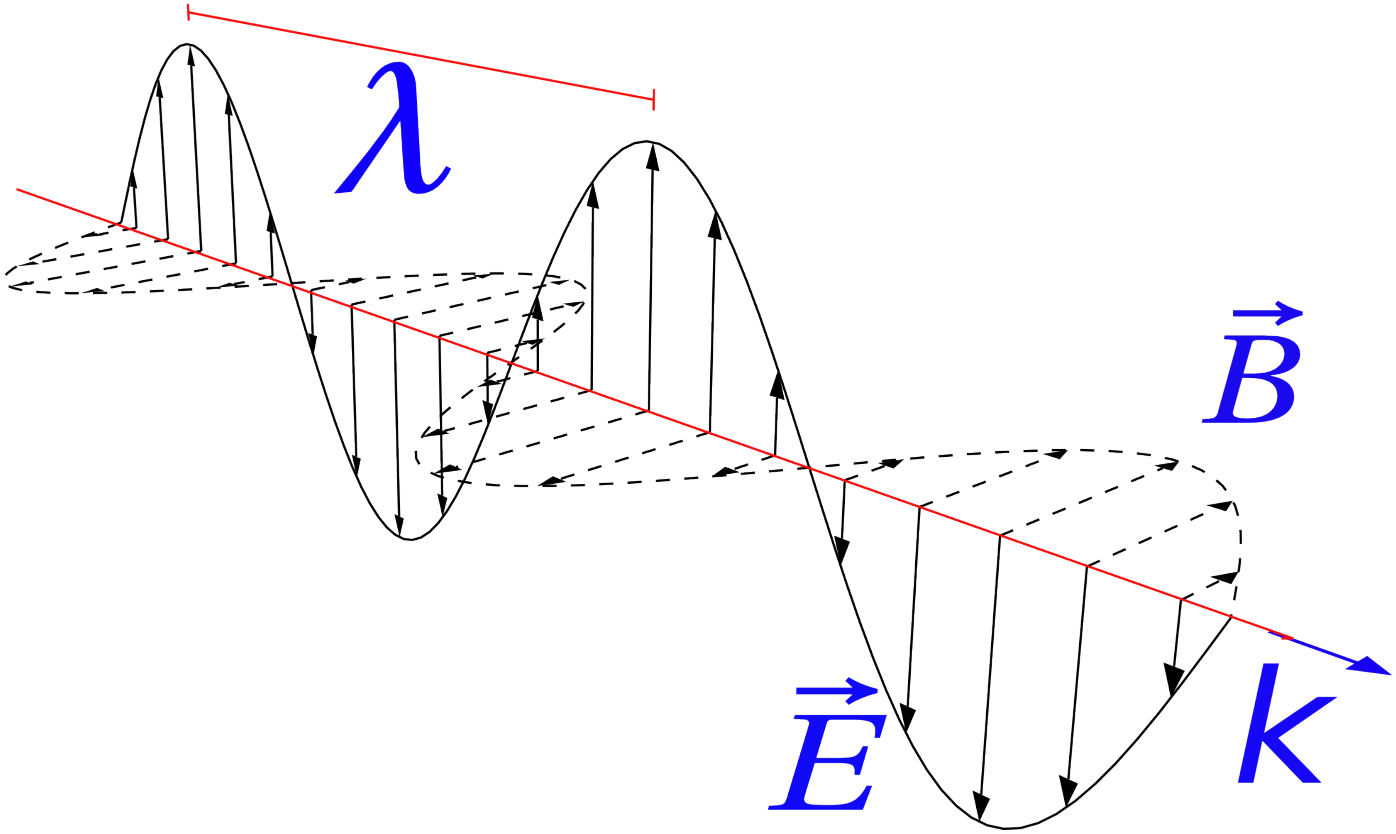
- In other words, the polarization vector $\hat{e}$ is perpendicular to the wavevector $\vec{k}$. This means that the electric field oscillates in direction perpendicular to the propagation direction.
- The same holds for the direction of oscillation of the magnetic field $\vec{B}$:

$$\vec{k}.\hat{b}A/c\exp[i(\vec{k}.\vec{r}-\omega t+p)]=0 \qquad \vec{k}.\hat{b}=0$$

- Fixing the electric field magnitude fixes the magnetic field amplitude. Only one degree of freedom here.
- Electric and magnetic fields are also perpendicular (and $\vec{k},\hat{e},\hat{b}$ constitute a right-hand oriented triple):

$$\hat{e}\perp\hat{b} \qquad \hat{b}=\frac{\vec{k}}{k}\times\hat{e} \qquad \hat{e}=-\frac{\vec{k}}{k}\times\hat{b}$$

- **linear polarization** occurs when $\hat{e}$ is real. At any location $\vec{r}$, the field then oscillates along a specific line. Other polarizations will be discussed later in the course.



Linearly polarized Electromagnetic Plane-Wave Geometry

Q: If a wave-vector $\vec{k}$ is given, what is the number of linearly independent polarizations vectors $\hat{e}$?

Q: A plane wave is characterized by a general wave-vector $\vec{k} = 2\pi/(800nm)\{0, 1/\sqrt{2}, 1/\sqrt{2}\}$ We also know that the electric polarization vector $\hat{e}$ has zero x-component.

A) Find $\hat{e}$

B) Calculate the magnetic polarization vector $\hat{b}$.

Q: Plane of polarization is one that is "spanned" by the wave-vector $\vec{k}$ and the electric field polarization vector $\hat{e}$.

What was the plane of polarization in the previous example?

Q: Show that given $\hat{e}$ and $\hat{b}$, the wave vector direction can be calculated as:

$$\frac{\vec{k}}{k} = \hat{e} \times \hat{b}$$

Transverse wave summary cont.:

Different ways to express phase argument in the exponential (or sin, cos):

- with wave-vector and angular frequency

$$\vec{k}.\vec{r} - \omega t + p$$

- with direction vector $\vec{n} = \vec{k}/k$, and wavelength

$$\frac{2\pi}{\lambda}\vec{n}.\vec{r} - \omega t + p$$

- with temporal frequency f

$$\frac{2\pi}{\lambda}\vec{n}.\vec{r} - 2\pi f t + p$$

- with temporal oscillation period T

$$\frac{2\pi}{\lambda}\vec{n}.\vec{r} - \frac{2\pi}{T}t + p$$

- with directional cosines

$$\frac{2\pi}{\lambda}(\cos\alpha_x x + \cos\alpha_y y + \cos\alpha_z z) - \frac{2\pi}{T}t + p$$

- ... and combinations, of course

Note: All these frequently appear in problem formulations and solutions...

Concrete plane-wave examples:

$$\vec{E}(\vec{r}, t) = \hat{j} E_{0y} \cos[kx - \omega t + p]$$

Electric field is:

- propagating along the x-axis
- linearly polarized along y-axis
- plane-wave, and independent of z and y

Concrete plane-wave examples:

$$\vec{E}(\vec{r},t) = \hat{j}E_{0y}\cos[kx - \omega t + p]$$

Electric field is:

- propagating along the x-axis
- linearly polarized along y-axis
- plane-wave, and independent of z and y

Magnetic field is:

- propagating along the x-axis
- linearly polarized along z-axis
- magnitude is E_{0y}/c
- $\hat{b} = +\hat{k}$ (no freedom for its sign!)

Electromagnetic plane waves in dielectric media

All derivations we have done can be repeated for dielectric medium instead of vacuum: The difference will be solely in

$$\epsilon_0 \to \epsilon_0 \epsilon_r$$

As a consequence, all we have established for plane waves holds with this simple replacement:

$$c \to \frac{c}{n(\omega)}$$

where $n(\omega)$ is the **index of refraction** for the particular angular frequency ω:

$$n(\omega)^2 = \epsilon_r(\omega)$$

Differences between wave propagation in dielectric and in vacuum:

- phase velocity decreases from $c/1$ to $c/n(\omega)$. Note that the refractive index is larger than one in most situations in transparent media. We will discuss this in more detail in the Section devoted to light-matter interactions.
- the in-medium dispersion relation now requires

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \frac{\omega^2 n(\omega)^2}{c^2} = \frac{\omega^2 \epsilon_r(\omega)}{c^2} = \frac{\omega^2 (1 + \chi(\omega))}{c^2}$$

- the wavelength in the medium is shorter by the factor given by the refractive index. This is why we sometimes emphasize which wavelength we mean by specifying "vacuum wavelength."
- consequently, as light passes from one medium to another, its wavelength in general changes, ...

Plane-wave wave-fronts:

Representing a harmonic wave trough its **phase**:

$$\vec{E}(\vec{r},t) = A\hat{e}\exp[i(\vec{k}.\vec{r} - \omega t + p)] = \vec{E}_0 e^{i\phi(\vec{r},t)}$$

Phase-fronts or **wave-fronts** are lines (surfaces in fact) of equal phase, e.g

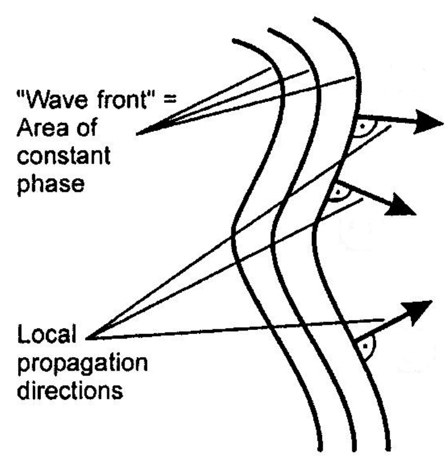
$$\phi(\vec{r},t) = 9.87654321$$

Note: that the above is a "moving" surface, in fact a plane (since we are talking about plane waves) that moves with the phase velocity in the direction of $\vec{k}$.

Note: superpositions of plane waves in general results in iso-phase surfaces that are not planes, but may have e.g. spherical shape.

Geometric optical rays: are paths that are everywhere orthogonal to phase fronts. Recalling from our math intro,

$$\vec{n}(\vec{r},t) \approx \nabla\phi(\vec{t},t)$$



For our harmonic solution we always have

$$\vec{n}(\vec{r},t) \approx \vec{k}$$

Energy density stored in the electromagnetic field

For a given "configuration" of the electromagnetic field, the local energy density stored in the field is given by:

$$U = \frac{1}{2}\mathbf{D}.\mathbf{E} + \frac{1}{2}\mathbf{B}.\mathbf{H}$$

For a non-magnetic, dielectric medium (i.e. $\mathbf{B} = \mu_0\mathbf{H}$) characterized by relative permittivity ϵ_r the energy density reduces to

$$U = \frac{1}{2}\epsilon_0\epsilon_r\mathbf{E}.\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}.\mathbf{B}$$

We have found that in a plane wave there is a definite relation between the electric and magnetic field magnitudes, namely

$$B^2 = \frac{1}{v^2}E^2 = \frac{n(\omega)^2}{c^2}E^2 = \epsilon_r(\omega)\epsilon_0\mu_0 E^2$$

Then

$$U = \frac{1}{2}\epsilon_0\epsilon_r E^2 + \frac{1}{2}\epsilon_0\epsilon_r E^2 = \epsilon_0\epsilon_r E^2$$

Note:

- The magnetic and electric contribution to the energy density in a plane wave are equal.
- Energy density is proportional to the square of the electric field magnitude. (Recall, that we have eluded to this property to interpret the $1/r$ factor in spherical waves as the one "taking care of energy conservation.")

Poynting vector and energy flow:

Now that we know the energy density, what is the energy flow?

Consider a volume

$$V = Avdt$$

given by area A perpendicular to the velocity of the wave v

Total energy transported in time dt is VU so the energy flow defined as amount of energy "crossing a surface" per unit area and unit time is then

$$S = \frac{VU}{Adt} = \frac{Adtv\epsilon_0\epsilon_r E^2}{Adt} = \frac{c}{n}\epsilon_0 n^2 E^2 = \epsilon_0 nc\mathbf{E}.\mathbf{E}$$

This is only magnitude of the flow, but it should have a direction (i.e. it should be a vector)!

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Check that **Poynting vector** $\vec{S}$ has the required properties:

- it has the same direction as the wave-vector: $\vec{S}||\vec{k}$
- it has the same magnitude as the expression S above:

$$|\vec{S}| = \frac{1}{\mu_0}EB = \frac{1}{\mu_0 v}E^2 = \frac{n}{\mu_0 c}E^2 = \frac{n\epsilon_0}{\epsilon_0\mu_0 c}E^2 = \frac{n\epsilon_0 c^2}{c}E^2 = \epsilon_0 ncE^2 = S$$

Time-averaged flow of energy

More explicitly, the magnitude of the Poynting vector is function of location in space and time:

$$S = \epsilon_0 nc\vec{E}(\vec{r},t).\vec{E}(\vec{r},t)$$

In a harmonic plane wave, it is

$$S = \epsilon_0 nc\vec{E}(\vec{r},t).\vec{E}(\vec{r},t) = \epsilon_0 ncE_0^2\cos^2[\vec{k}.\vec{r}-\omega t]\ ,$$

which value oscillates between zero and a maximal value $\epsilon_0 ncE_0^2$: energy propagates in "chunks."

$$S = \epsilon_0 ncE_0^2\left(\frac{1}{2}+\frac{1}{2}\cos[2(\vec{k}.\vec{r}-\omega t)]\right)$$

The second term if **fast**, and a detector will average it to zero. What remains is the time-averaged Poynting vector:

$$< S >_T = I = \frac{1}{2}\epsilon_0 ncE_0^2 \quad \text{unit :} \quad \frac{W}{m^2}$$

I is called **irradiance** or **intensity**.

Note: We have "derived" Poynting specifically for a plane wave situation, but the formula $\vec{S} = \vec{E}\times\vec{H}$ is valid in general.

Classical vs Quantum: Conceptual differences

Classical:

- particle has a sharp position and velocity
- measured values are "selected" from continuum
- measured values are sharply defined
- measured values do not affect each other
- measurement does not disturb the system

Quantum:

- particle can not have a sharp position and velocity *at the same time*
- measured values are "selected" from continuum or from a discrete sets
- measured values may be "blurred," each individual measurement is random, *it is the average we measure*
- measurement changes the system's state
- the same object may behave as a discrete particle *and* as a wave

Photons: Quantum nature of light

Classical theory:

$$I = \frac{1}{2}\epsilon_0 nc|\vec{E}_0|^2$$

Quantum theory:

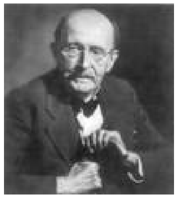
$$I = \frac{1}{\text{area}}\frac{1}{\Delta\text{time}}\sum_{photons} E_{ph}$$

In other words, energy transported by a beam is not infinitely divisible. There is a minimal quantum, called **photon** that carries the energy

$$E_{ph} = \hbar\omega = h\nu \qquad \hbar = \frac{h}{2\pi}$$

Where:

- **Planck constant** is $\hbar = 1.054571726(47) \times 10^{-34}$ Js



Planck, Max
1858 — 1947

- this energy quantum is proportional to the frequency of light
- its magnitude is really small on the human scale
- most light sources emit zillions of photons

Light pressure

Besides energy quantum, also momentum is divided between photons:

$$\vec{p} = \hbar\vec{k} = \frac{h}{\lambda}$$

Note:

- the direction is that of wave propagation
- $\vec{k}$ plays a similar role as a classical particle velocity ($\vec{p}_{classical} = m\vec{v}$)

Consequence: Light exerts force, or pressure.
Classical picture of light pressure: Pressure equals energy density

$$\mathcal{P} = U = \frac{S(t)}{c}$$

... averaged over optical cycle:

$$< \mathcal{P} >_T = \frac{I}{c}$$

Quantum picture of light pressure:

Force × time = change in momentum = number of absorbed photons × momentum of each

$$A\mathcal{P}\Delta t = F\Delta t = A\ \Phi\ \Delta t\ \hbar|\vec{k}|$$

where A is area, and Φ is the **photon flux density**, i.e. number of photons (hitting the target) per unit time, per unit of area.

So the connection between classical and quantum is:

$$\mathcal{P} = \Phi\hbar k = \Phi\frac{h}{\lambda} \qquad I = \Phi\hbar\omega \qquad \mathcal{P} = \hbar k\frac{I}{\hbar\omega} = \frac{I}{c}$$

Note: In the above, we have assumed that all photons were completely absorbed. The situation changes when they would be reflected: The net change of momentum per photon would double and, consequently the pressure or force would also double.
