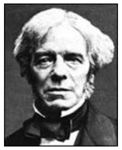
Underpinnings of the EM theory:

1. Forces of two kinds, electric and magnetic, acting on charged particles

Lorentz, Hendrik Antoon
1853-1928

2. A time-varying magnetic field has an electric field associated with it



Faraday, Michael
1791-1867

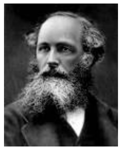
3. Flux of EM field is zero from a charge-free region

Gauss, Carl Fridrich
1777-1855

4. A time-varying electric field has an associated magnetic field

Ampere, Andre Marie
1775-1836

5. Displacement current "closes" seemingly open electrical circuits



Maxwell, James Clerk
1831-1879

Lorentz force law

A charged particle of charge q, moving with velocity $\vec{v}$ through *electric* and *magnetic* fields $\vec{E}$ and $\vec{B}$ experiences a force given by:

$$\vec{F} = \vec{F}_E + \vec{F}_M = q\vec{E} + q\vec{v} \times \vec{B}$$

Mathematically, $\vec{E}$ and $\vec{B}$ are vector fields, e.g. $\vec{B} = \vec{B}(\vec{r}, t)$

Effect of EM fields on charged particles are the *only way* we can "know" these fields.

Static $\vec{E}$ field effect (in $\vec{B} = 0$)

$$\vec{F}_E \quad || \quad \vec{E}$$

Static $\vec{B}$ field effect (in $\vec{E} = 0$)

$$\vec{F}_M \quad \perp \quad \vec{B} \quad , \quad \vec{F}_M \quad \perp \quad \vec{v}$$

Thus: relation between acceleration and velocity vector tells us whether we deal with the magnetic or electric field.

Gauss law:

The ultimate source of all EM fields is the motion of charged particles, e.g.

- Laser — electrons in atoms/molecules/quantum wells/...
- Antenna — electrons in wires
- Cyclotron — accelerated charges

In a volume with no charges there is no source for the fields and it follows that the flux of Em fields will be zero:

$$\Phi_E = \iiint_V dV \nabla.\vec{E} - \iint_S d\vec{S}.\vec{E} = 0$$

electric:

magnetic:

$$\Phi_M = \iiint_V dV \nabla.\vec{B} - \iint_S d\vec{S}.\vec{B} = 0$$

where it should be understood that S is the closed surface of the volume denoted by V.

There is no evidence of magnetic charges (monopoles), and the magnetic Gauss law holds as written.
If the volume contains a total charge $\sum q$, then the electric flux is

$$\Phi_E = \iint_S d\vec{S}.\vec{E} = \frac{1}{\epsilon_0}\sum q = \frac{1}{\epsilon_0}\iiint_V dV\rho$$

This is the Gauss law for the electric field, with

$$\epsilon_0 = 8.854 \times 10^{-12} F/m \quad As/Vm \quad C^2/N.m^2$$

standing for the permittivity of free space.
The above is still not the most general case: we have tacitly assumed that charges are embedded in vacuum. However, the integration surface may be in a medium. In such a case

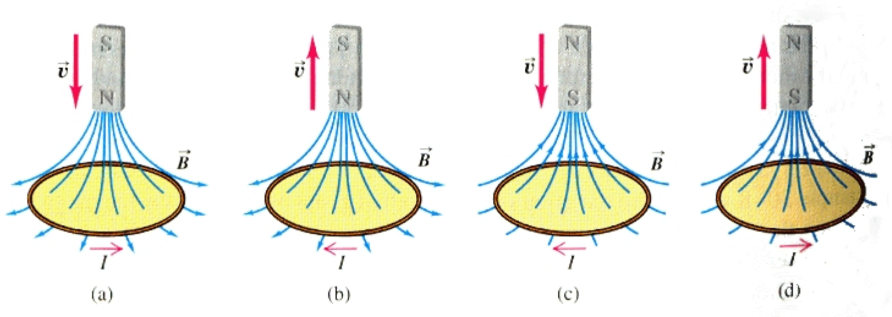
$$\epsilon_0 \to \epsilon_r \epsilon_0 \equiv \epsilon$$

where ϵ_r is the dielectric "constant" or relative permittivity of the host medium, and for $\vec{D} = \epsilon\vec{E}$:

$$\iint_S d\vec{S}.\vec{D} = \sum q = \iiint_V dV\rho$$

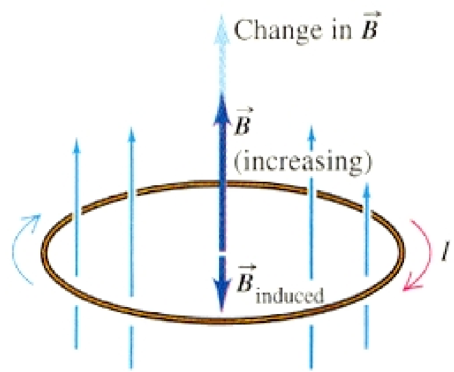
Faraday's law:

$$\oint_C \vec{E}.\vec{dl} = -\frac{d}{dt}\iint_A \vec{B}.d\vec{S}$$



Lenz's law:

Lentz, Heinrich
1804 - 1865

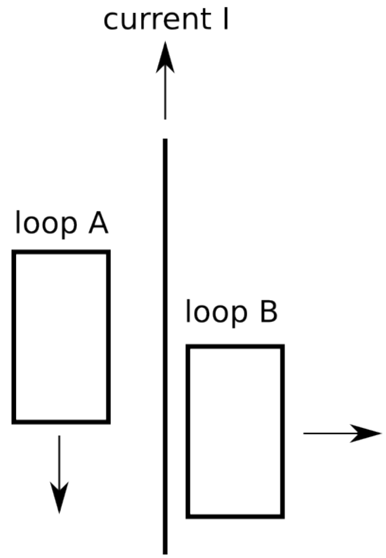


Current induced by change in magnetic flux generates additional field that always opposes the original change.

Note: Use this to figure out directions of currents, fields, forces, ...

Problem:

A) Using the Lenz's law, determine the orientation of current induced in the loops when they are pulled in the directions indicated:



Hint: First, figure out the direction of the magnetic field piercing the loops. Then think about how the flux <u>changes</u> with the movement, and finally come up with a current which would generate field with opposite orientation. Keep in mind that is the the **change** of the flux that you are to consider.

B) Now utilize Lorenz force to figure out the direction of induced currents.

Hint: Force that is perpendicular to a wire "does not count."

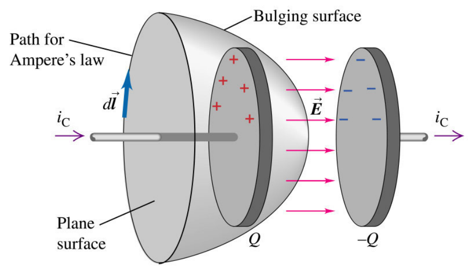
Ampere's law:

Consider first a static magnetic field:

$$\oint_C \vec{dl}.\vec{B} = \mu_0 \sum i = \mu_0 \iint_S \vec{J}.d\vec{S}$$

The line integral of $\vec{B}$ along a closed curve C is the total of current that crosses the surface bounded by C.

However, this does not hold if time-varying electric field is present!



James Clerk Maxwell corrected this by postulating the "displacement current density"

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$$

that is added to the current density $\vec{J}$:

$$\oint_C \vec{dl}.\vec{B} = \mu_0 \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) .d\vec{S}$$

Electromagnetic fields:

- **E** Electric field intensity, in V/m or $kgms^{-3}A^{-1}$ in base units
- **H** Magnetic field strength, or intensity, in A/m
- **D** Electric displacement field, electric induction, in As/m^2
- **B** Magnetic field, magnetic flux density, magnetic induction, in Vs/m^2 or $kgs^{-2}A^{-1}$ in base units
- ϵ_0 permittivity, in $F/m = As/Vm$ or $kg^{-1}m^{-3}s^4A^2$ in base units
- μ_0 permeability, in $H/m = Vs/Am$ or $kgms^{-2}A^{-2}$

- **P** Polarization, or dipole-moment density in a medium
- **M** Magnetization, or magnetic moment density in a medium

Why two fields instead of one?

Electric field "pair:" **E** and **D**
Magnetic field "pair:" **B** and **H**

Physical meaning:
1. Lorentz force gives meaning to **E**, **B**:

$$\vec{F} = q\mathbf{E} + q\vec{v} \times \mathbf{B}$$

2. Electric induction, D-field, displacement field, ..., **D** and magnetic field intensity **H**:
They can be understood as part of the electric and magnetic field that are "independent" of the medium. This is the part of the el-mag field we create by controlling charges and currents.

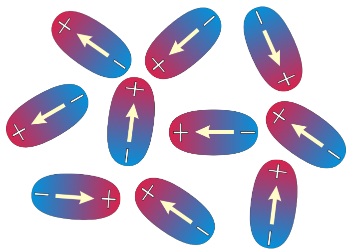
Example: In a capacitor with a given charge, **D** between plates (electrodes) does not depend on the dielectric medium filling between the electrodes. The electric field **E** does:

$$|\mathbf{D}| = \frac{Q}{S} \qquad |\mathbf{E}| = \frac{1}{\epsilon_0 \epsilon_r} \frac{Q}{S}$$

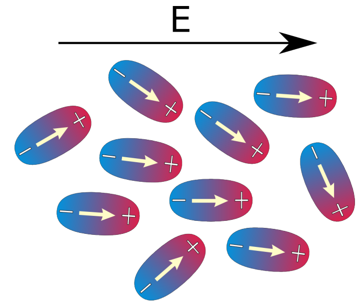
Example: Inside a solenoid, magnetic field intensity **H** does not depend on what material is the core of the electromagnet made of. In contrast, magnetic field **B** does:

$$|\mathbf{H}| = \frac{NI}{L} \qquad |\mathbf{B}| = \mu_0 \mu_r \frac{NI}{L}$$

Polarization = dipole moment density in a medium exposed to field



uncorrelated dipoles result in zero average dipole moment



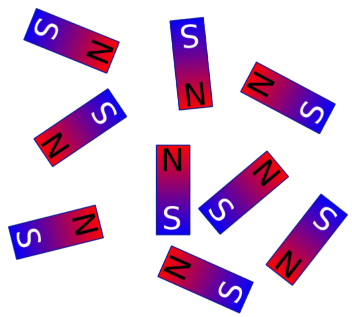
when the electric field partially orders the dipole, the net dipole moment is nonzero

$$\mathbf{P} = \frac{\sum \vec{d}_i}{V} \quad , \quad \vec{d}_i = q(\vec{r}_i^{\,+} - \vec{r}_i^{\,-})$$

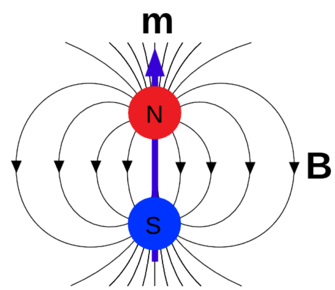
$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(\{\mathbf{E}\})$$

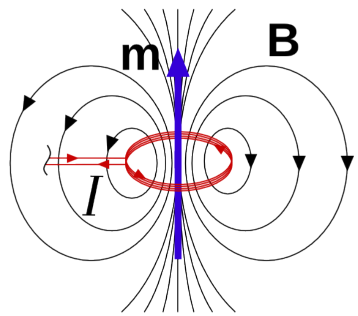
In the Section on light-matter interactions, we will explore ways to calculate the polarization term above...

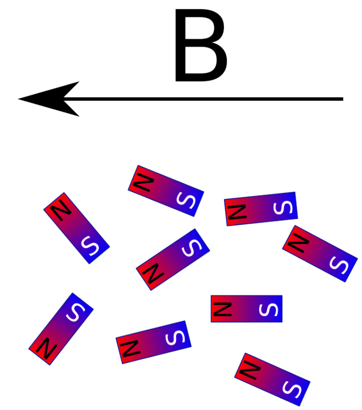
Magnetization = magnetic moment density in a medium exposed to field



uncorrelated "magnets" result in zero average magnetic moment







when the magnetic field partially orders the "magnets", the net magnetic moment is nonzero

$$\mathbf{M} = \frac{\sum \vec{m}_i}{V} \quad , \quad \vec{m}_i = \frac{1}{2}\int \vec{r} \times \vec{J} dV$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H}))$$

In this course we restrict our attention to non-magnetic materials, in which $\mathbf{M} = 0$.

Dielectric constant:

- it is not constant! Q: Function of what quantities is it?
- it its an expression of how the electric field permeates the material
- in view of the above "elementary dipole" picture, it measures the ability of the medium to respond to the external field by ordering its elementary dipoles. But the dipoles need not be permanent!
- most often characterized by <u>relative permittivity</u> ϵ_r
- also used is <u>susceptibility</u> χ

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \equiv \epsilon_0 \epsilon_r \mathbf{E} \equiv \epsilon_0 (1 + \chi) \mathbf{E} \qquad \mathbf{P} = \epsilon_0 \chi \mathbf{E}$$

Soon we will see that it determines refractive index of the material:

$$n^2 = \epsilon_r = 1 + \chi$$

Maxwell equations summary

Differential form

Ampere and Faraday

$$\begin{aligned} \nabla \times \mathbf{H} &= +\partial_t \mathbf{D} + \mathbf{J} \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \end{aligned}$$

Integral form

Ampere and Faraday

$$\oint_C \mathbf{B}.\vec{dl} = \iint_S (\nabla \times \mathbf{B}) .d\vec{S} \;=\; \mu_0 \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) .d\vec{S}$$

$$\oint_C \mathbf{E}.\vec{dl} = \iint_S (\nabla \times \mathbf{E}) .d\vec{S} \;=\; - \iint_S \left(\frac{\partial \mathbf{B}}{\partial t} \right) .d\vec{S}$$

Gauss, electric and magnetic

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{D} &= \rho \end{aligned}$$

$$\iint_S \mathbf{B}.d\vec{S} \;=\; \int_V \nabla.\mathbf{B} dV = 0$$

$$\iint_S \mathbf{D}.d\vec{S} \;=\; \int_V \nabla.\mathbf{D} dV = \int \rho dV$$

Medium (constitutive) relations, source terms

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} \\ \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H} \end{aligned}$$

$$\begin{aligned} \mathbf{P} &= \mathbf{P}(\{\mathbf{E}(t)\}) \\ \mathbf{J} &= \mathbf{J}(\{\mathbf{E}(t)\}) \end{aligned}$$

Charge conservation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0$$

Maxwell equations summary (specialized for non-magnetic medium)

$$\begin{aligned}\nabla \times \mathbf{H} &= +\partial_t \mathbf{D} + \mathbf{J} \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \quad , \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(\{\mathbf{E}\}) = \epsilon_0 \epsilon_r \mathbf{E}$$

This is the form we will use in Section on light-matter interaction...

Maxwell equations summary (specialized for vacuum)

$$\begin{aligned} \nabla \times \mathbf{B} &= +\frac{1}{c^2}\partial_t \mathbf{E} \qquad \frac{1}{c^2} \equiv \epsilon_0 \mu_0 \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

This is the form we will start from (to explore properties of EM waves) next ...

Exercise: Check that dimensions on both sides of $\epsilon_0 \mu_0 = 1/c^2$ agree.

It is useful to remember these values:

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\ H/m \ \ (H/m = Vs/Am) \\ \epsilon_0 &= 8.854\ldots \times 10^{-12}\ F/m \ \ (F/m = As/Vm) \\ c &= 299,792,458\ m/s \end{aligned}$$

Note: These values are exact.

Maxwell equation summary (specialized for harmonic time-dependent fields)

Often (and especially in OPTI-310) it is sufficient to consider harmonic time-dependence in the electromagnetic fields.
Using our complex representation, we can look for solutions with the Ansatz

$$\mathbf{E}(\vec{r},t) = \mathbf{E}(\vec{r})e^{-i\omega t} \quad \mathbf{B}(\vec{r},t) = \mathbf{B}(\vec{r})e^{-i\omega t} \quad ,$$

... and recall the operator "equivalencies" discussed previously. We get

$$\begin{aligned} \nabla \times \mathbf{B} &= +\frac{1}{c^2}(-i\omega)\mathbf{E} \\ \nabla \times \mathbf{E} &= -(-i\omega)\mathbf{B} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

The advantage of this formulation is that a problem reduces from "time-and-space" to "space" only

Note: The same Ansatz can be used for ME in media. Such a formulation often used for, e.g., calculation of waveguide properties.

Question:

Consider an initial-value problem. $\mathbf{E}$ and $\mathbf{B}$ are given for $t = 0$.

Q: How many independent equations there are to solve? We have six components between, say, $\mathbf{E}$ and $\mathbf{B}$. Having satisfied Ampere and Faraday, it may seem that we are left with no degrees of freedom to also satisfy Gauss? So, are there too many equations to be satisfied simultaneously!?

Divergence equations as Initial Constraints

Here we show that the Gauss law equations represent only constraints on the initial condition for a solution of Maxwell equations.

Consequence: We only need to worry about $\nabla.$ equations at time $t = 0$. Afterward they are satisfied automatically as a consequence of:

- Ampere
- Faraday
- charge conservation

Assume that $\nabla \cdot \mathbf{D}(t = 0) = \rho(t = 0)$. If we show that $\partial_t(\nabla \cdot \mathbf{D}(t) - \rho(t)) = 0$ for all times, it follows that $\nabla \cdot \mathbf{D}(t) = \rho$ for all t.

$$\partial_t \nabla \cdot \mathbf{D}(t) = \nabla \cdot \partial_t \mathbf{D} = \nabla \cdot (\nabla \times \mathbf{H} - \mathbf{J}) = -\nabla \cdot \mathbf{J}$$

and therefore

$$\partial_t(\nabla \cdot \mathbf{D}(t) - \rho(t)) = -\nabla \cdot \mathbf{J} - \partial_t \rho(t) = 0 \ .$$

Exercise: Show similar argument for the magnetic Gauss.