Single-slit (Fraunhofer) diffraction

Apply HFP:

What we need to do is to add up (i.e. integrate) contributions from all secondary sources (of spherical waves) in the slit:

$$E = E_L \int_{SLIT} \frac{e^{ikr}}{r} dy$$

As discussed earlier, we utilize the linear (in y) approximation of r:

$$r \approx R - y \sin\theta$$

in the exponential, and the crudest (zero-order) in the denominator

$$r \approx R$$

We thus have:

$$E = E_L \int_{-b/2}^{+b/2} \frac{e^{ik(R-y\sin\theta)}}{R} dy \qquad = E_L \frac{e^{ikR}}{R} \int_{-b/2}^{+b/2} e^{-iky\sin\theta} dy$$

$$E = E_L \frac{e^{ikR}}{R} \frac{(e^{-i\beta} - e^{+i\beta})}{-ik\sin\theta} \qquad \beta = \left(\frac{kb}{2}\right) \sin\theta$$

and finally

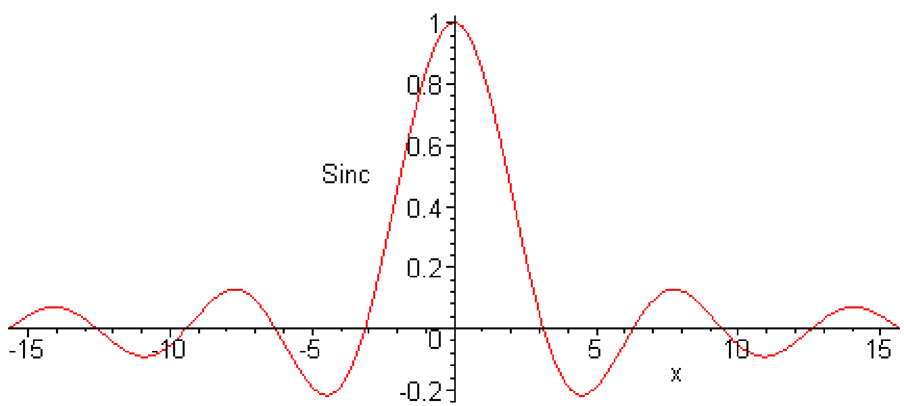
$$E = E_L \frac{be^{ikR}}{R} \frac{\sin\beta}{\beta}$$

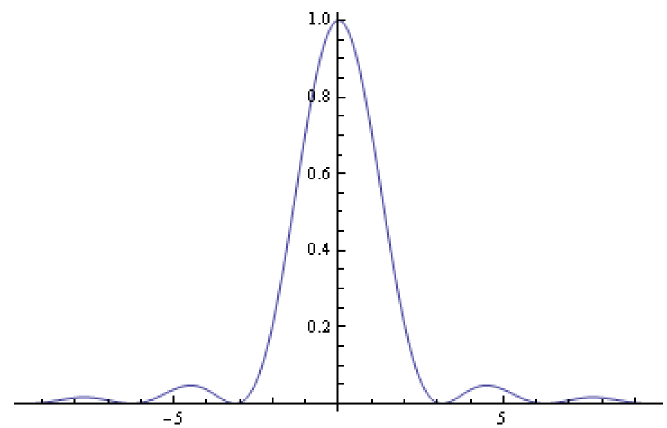
Contributions to the diffraction amplitude:

$$E = E_L \frac{be^{ikR}}{R} \frac{\sin\beta}{\beta} \qquad \beta = \left(\frac{kb}{2}\right)\sin\theta$$

- Overall laser-amplitude
- scaled spherical wave amplitude
- modulation due to aperture: $\sin\beta/\beta \equiv \mathrm{sinc}\beta$

Note: β measures the "position" in the far field (it relates to the angle θ. It also sets the scale in the far field: This is determined by the **width of the slit**.





Fringe locations

$$\frac{\sin(\beta)}{\beta} = \mathrm{sinc}(\beta) = \mathrm{sinc}\left(\frac{kb}{2}\sin\theta\right)$$

$$\frac{I(\theta)}{I(0)} = \mathrm{sinc}^2\left(\frac{kb}{2}\sin\theta\right)$$

In the Fraunhofer region, angle θ must be small, therefore $\sin\theta \sim \theta$. Zeros occur at

$$\frac{kb}{2}\theta_m \sim m\pi \quad , \quad m = \pm 1, \pm 2, \ldots$$

or

$$\theta_m = m\left(\frac{\lambda}{b}\right)$$

If observed on a screen (distance x), transverse location y_m of the m-th null obeys

$$\tan\theta_m = \frac{y_m}{x} << 1 \quad \text{or} \quad y_m \sim x\theta_m = mx\frac{\lambda}{b}$$

Fraunhofer diffraction on a single slit summary:

- pattern given by the sinc function
- pattern expands with the distance from the screen
- zeros and maxima along fixed angles of propagation
- both spatial and angular scale of the pattern scales with the ratio $\frac{\lambda}{b}$
- characteristic scale (i.e. feature size) in the pattern inversely proportional to the characteristic length-scale of the aperture function
- far-field pattern *amplitude* is essentially a Fourier transform of the aperture function

$$E = E_L \frac{e^{ikR}}{R} \int_{APERTURE} e^{-ik_y y} dy \qquad k_y = k \sin\theta$$

Interpretation in terms of plane waves

Single plane wave $\times$ Aperture $=$ Superposition of plane waves

Far field intensity $=$ Amplitude^2 as a function of the plane-wave propagation angle

This problem illustrates certain universal properties of far-field (i.e. Fraunhofer) diffraction patterns.

P0: Consider 1D Fraunhofer diffraction through a slit. Following the derivation procedure used in the class,

A) Show what happens to the diffraction pattern when we use a plane wave that is incident not perpendicular but at angle γ on the screen.

B) What happens if the slit is moved up by a certain shift s? Is there a change visible in the intensity of the far field pattern?

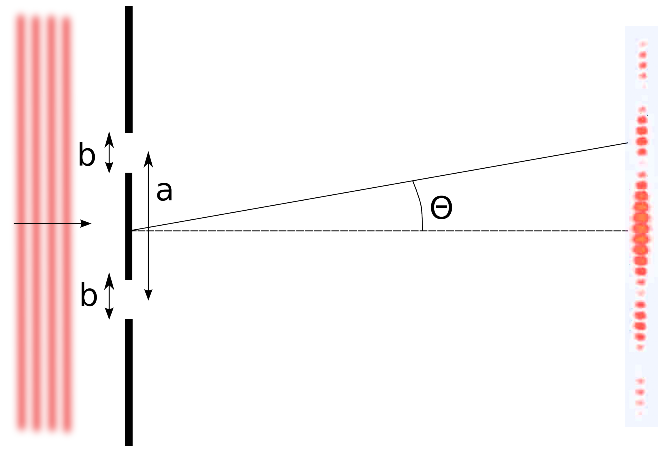
C) Show that if the incident plane wave is normal to the screen, then the pattern has an inversion symmetry: If the angle in the far field is θ then the intensity is the same at $I(\theta)$ as for the opposite angle,

$$I(\theta) = I(-\theta)$$

D) This result may seem trivial for a single slit. Generalize it for an arbitrary collection of slits, for example for a pair with one wide and one narrow slit. In other words, show that even if the screen does not have up-down symmetry, the diffraction pattern does.

Note: We will later see that this symmetry property remain true also in 2D diffraction: Instead of up-down symmetry, we will speak about *inversion* symmetry.

Two-slit diffraction

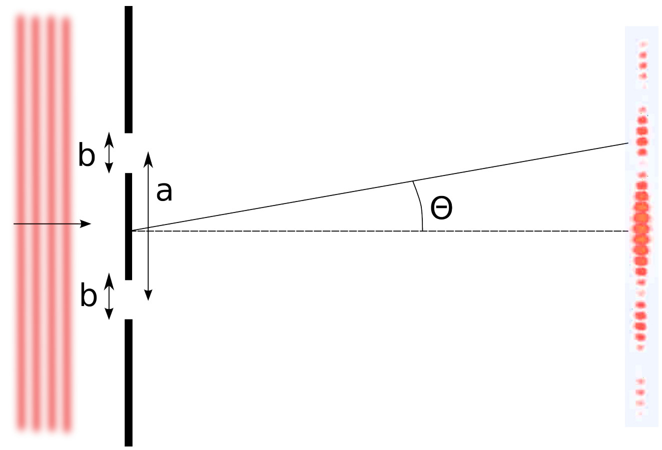


Solution A: Using a superposition principle, together with one result from the problem above. The far-field amplitude is a sum of amplitudes from each slit. Each slit amplitude is modified by a phase shift corresponding to a spatial shift $\pm a/2$:

$$E = E_U + E_L = E_{slit} e^{-ia/2k\sin\theta} + E_{slit} e^{+ia/2k\sin\theta} = 2E_{slit}\cos\left(ka/2\sin\theta\right)$$

$$E = 2E_0\left(\frac{be^{ikR}}{R}\right)\mathrm{sinc}(\beta)\cos(\alpha) \qquad \beta = \frac{kb}{2}\sin\theta \qquad \alpha = \frac{ka}{2}\sin\theta$$

Two-slit diffraction



Solution B: Honest calculation. Using HFP.

$$E = E_0 \frac{e^{ikR}}{R} \int_{+a/2-b/2}^{+a/2+b/2} dy e^{-iky\sin\theta} + E_0 \frac{e^{ikR}}{R} \int_{-a/2-b/2}^{-a/2+b/2} dy e^{-iky\sin\theta}$$

Performing the integrals leads to the same result:

$$E = 2E_0 \left(\frac{be^{ikR}}{R}\right) \mathrm{sinc}(\beta)\cos(\alpha) \qquad \beta = \frac{kb}{2}\sin\theta \qquad \alpha = \frac{ka}{2}\sin\theta$$

Intensity pattern:

$$I(\theta) = 4I(0)\mathrm{sinc}^2(\beta)\cos^2(\alpha)$$

Double-slit diffraction summary

- the resulting pattern is a “product” of a slit-pattern and a Young‘s double-source pattern
- each characteristic dimension (a, b) shows up in the far field as a modulation with spatial frequency inversely proportional to that dimension

N-slit Fraunhofer diffraction

$$E = \sum_{s=0}^{N-1} E_{slit} e^{-isak\sin\theta} = E_{slit}\frac{1-e^{-iNak\sin\theta}}{1-e^{-iak\sin\theta}} = E_{slit}\frac{e^{-iNak/2\sin\theta}}{e^{-iak/2\sin\theta}}\frac{e^{+iNak/2\sin\theta}-e^{-iNak/2\sin\theta}}{e^{+iak/2\sin\theta}-e^{-iak\sin\theta}}$$

$$E = E_{slit} e^{-i(N-1)\frac{ak}{2}\sin\theta}\,\frac{\sin\left(\frac{Nak}{2}\sin\theta\right)}{\sin\left(\frac{ak}{2}\sin\theta\right)}$$

the total amplitude = shift-related phase $\times$ single slit $\times$ N-slit modulation

$$E = e^{-i(N-1)\frac{ak}{2}\sin\theta}\,\frac{\sin\beta}{\beta}\,\frac{\sin N\alpha}{\sin\alpha}$$

intensity pattern:

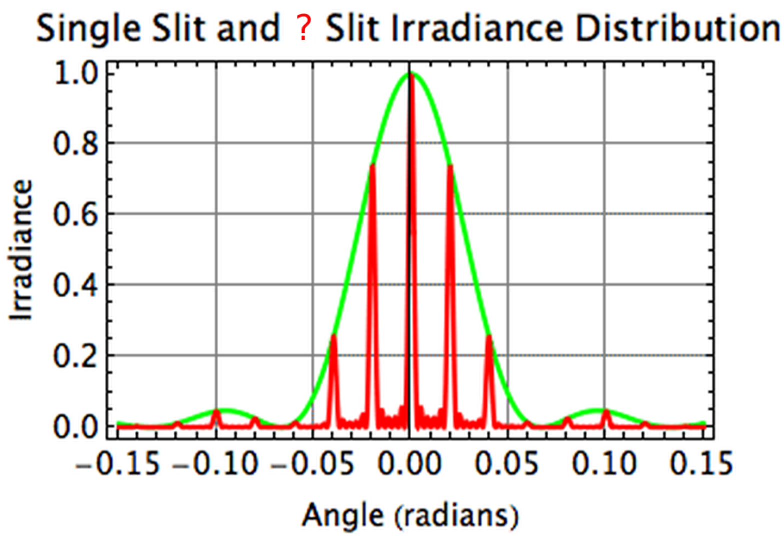
$$I(\theta) = \left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

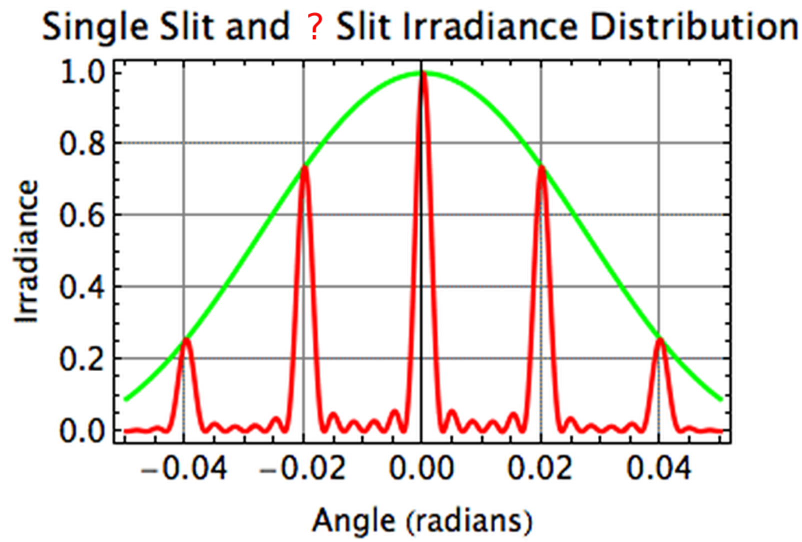
(spatial shift not observable in the far-field diffraction *intensity* pattern)

Note: There are three scales in this: b, a and Na. They all show up in the diffraction pattern.

N-slit Fraunhofer diffraction

From J. Wyant website:





Exercise: Estimate the parameters of the N-slit from the above plots. Express both the slit width b and slit spacing a in units of wavelength.