Problem Example:

Consider a slab of negative uni-axial crystal, thickness L, with the optic axis parallel to its surface, and refractive indices n_o and n_e. An electromagnetic plane wave is incident from vacuum, in direction normal to the surface of the slab. Given are:

- the electric field amplitude E_0
- wavelength λ
- linear polarization direction 45 deg w.r.t. optic axis
- also assume that no reflections occur at material interfaces.

A) Write the complex representation of the incident plane wave solution

B) Find the plane wave solution inside the crystal. You should observe that the polarization of the wave depends on the propagation distance.

C) Find the condition that the thickness L has to satisfy in order to have the outgoing plane wave polarized at 90 deg w.r.t. the incident wave polarization.

Solution:

General procedure (for propagation in anisotropic media):

1. Split the incident wave into ordinary and extraordinary
2. Propagate each separately through the medium, with their respective refractive indices
3. Recombine the two waves into one, and propagate outside

Step 1.

Choose frame of reference: Propagation along y, optic axis along z.

A) Wave outside of the slab, for $y < 0$:

$$\vec{E}_{before}(y,t) = \frac{\hat{i} + \hat{k}}{\sqrt{2}} E_0 \exp[i\frac{2\pi}{\lambda}y - \omega t]$$

B) Wave inside, for $0 < y < L$, with the help of so-far unknown amplitudes A_o and A_e:

$$\vec{E}_{inside}(y,t) = \vec{E}_o + \vec{E}_e \equiv A_o\hat{i}\exp[i\frac{2\pi n_o}{\lambda}y - \omega t] + A_e\hat{k}\exp[i\frac{2\pi n_e}{\lambda}y - \omega t]$$

Continuity at $y = 0$ requires

$$\vec{E}_{inside}(y=0,t) = \vec{E}_{before}(y=0,t)$$

which is satisfied when

$$A_o = 1/\sqrt{2} \qquad A_e = 1/\sqrt{2}$$

In words: We represent the inside wave as a sum of ordinary plus extraordinary with unknown amplitudes and find these amplitudes by joining the inside and outside solutions smoothly.

Step 2:

This step is almost trivial. Having the two-wave representation, "propagation" will simply "insert" $y = L$ into our formula:

$$\vec{E}_{inside}(y = L, t) = \frac{\hat{i}}{\sqrt{2}} \exp[i\frac{2\pi n_o}{\lambda} L - \omega t] + \frac{\hat{k}}{\sqrt{2}} \exp[i\frac{2\pi n_e}{\lambda} L - \omega t]$$

Step 3:

Wave after the slab must be an ordinary plane-wave, but we do not know its vector amplitude $\vec{a}$:

$$\vec{E}_{after}(y, t) = \vec{a} E_0 \exp[i\frac{2\pi}{\lambda}(y - L) - \omega t] \quad , \quad y \leq L$$

This must smoothly join the inside solution:

$$\vec{E}_{inside}(y = L, t) = \vec{E}_{after}(y = L, t)$$

and this requires that

$$\vec{a} = \frac{1}{\sqrt{2}} \left(\hat{i} \exp[i\frac{2\pi n_o}{\lambda} L] + \hat{k} \exp[i\frac{2\pi n_e}{\lambda} L] \right) = \frac{1}{\sqrt{2}} \left(\hat{i} \exp[i\frac{2\pi(n_o - n_e)}{\lambda} L] + \hat{k} \right) \exp[i\frac{2\pi n_e}{\lambda} L]$$

C) For $\vec{a}$ to correspond to an oscillation direction rotated by 90 degrees, we need to have

$$\vec{a} \sim -\hat{i} + \hat{k}$$

which means that we need to ask for

$$\exp[i\frac{2\pi(n_o - n_e)}{\lambda}L] = -1$$

which we solve for L to obtain:

$$\frac{2\pi(n_o - n_e)}{\Lambda}L = \pi k \qquad k = 1, 2, 3, \ldots$$

The smallest L solution (obtained for $k = 1$) then has the property that

$$(n_o - n_e)L = \frac{\lambda}{2}$$

In words, the difference in optical lengths of the crystal between the ordinary and extraordinary rays is equal to one half of the wavelength.