Fresnel diffraction at a straight edge

Here we look at the "universal" structure of the boundary between illuminated regions and the shade.

Let the screen is given by

$$x = 0 \qquad z < 0$$

and a plane wave of wavelength λ is incident from $x = -\infty$. Our observation point P is

$$P = (D_p, 0, d)$$

The question is what is the intensity observed at P as a function of d?

Solution:

Set up the integral over the "aperture" that is now an infinite half-plane. The incident amplitude is equal to one everywhere, and we neglect the spherical wave factor.

$$u_p \approx \int_{-\infty}^{+\infty} dy \int_0^{+\infty} dz \; 1 \; \exp\left[+ik\sqrt{D_p^2 + y^2 + (z-h)^2}\right]$$

To neglect the attenuation of the amplitude on its travel from the point in the aperture to P, we assume that the integral will be dominated by the small area close to P over which the variation of $1/R$ is "small." We are not going into details of justification...

Next we we perform the same approximation in the distance calculation as before when we discussed the difference between Fraunhofer and Fresnel: We keep terms of the second order.

$$\sqrt{D_p^2 + y^2 + (z-h)^2} \approx D_p + \frac{y^2 + (z-d)^2}{2D_p}$$

In this approximation we have to evaluate

$$u_p \approx \int_{-\infty}^{+\infty} dy \int_0^{+\infty} dz \; 1 \;\; \exp\left[+ik\left(D_p + \frac{y^2 + (z-d)^2}{2D_p}\right)\right]$$

This factorizes into a product of three terms:

$$u_p \approx e^{ikD_p} \int_{-\infty}^{+\infty} dy \exp\left[+ik\frac{y^2}{2D_p}\right] \int_0^{+\infty} dz \exp\left[+ik\frac{(z-d)^2}{2D_p}\right]$$

We can throw away the first two. They are un-interesting because they are not functions of d. What remains is the integral that can not be evaluated in terms of elementary functions:

$$u_p \approx \int_0^{+\infty} dz \exp\left[+ik\frac{(z-d)^2}{2D_p}\right]$$

It is useful to transform

$$u_p \approx \int_0^{+\infty} dz \exp\left[+ik\frac{(z-d)^2}{2D_p}\right]$$

into a more "universal" form by substitution

$$\eta^2 = k\frac{(z-d)^2}{2D_p}$$

$$u_p \approx \int_{-w}^{+\infty} d\eta e^{i\eta^2} \qquad w = d\sqrt{\frac{k}{2D_p}}$$

Even if we could not find any usable information about the above integral (and that is not the case, of course), what we have here is already a non-trivial result. It tells us on what spatial scale is the shadow turning into illuminated region. This (inverse) scale is given by factor next to d:

$$\sqrt{\frac{2D_p}{k}} = \sqrt{\frac{D_p\lambda}{\pi}} \approx \sqrt{D_p\lambda}$$

The result

$$u_p \approx \int_{-w}^{+\infty} d\eta e^{i\eta^2} \qquad w = d\sqrt{\frac{k}{2D_p}}$$

can be expressed in terms of well-studied Fresnel integrals

$$C(z) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} \cos \eta^2 d\eta \qquad C(\infty) = 1/2$$

$$S(z) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} \sin \eta^2 d\eta \qquad S(\infty) = 1/2$$

(Note that this is not the only common normalization of these functions.)

We are not going to utilize these functions. For our purposes, it will suffice to understand main features of the function we have derived...

1. Intensity at the geometric light-shadow boundary

This is the easiest case. First consider $d \to \infty$ which means an observation point deep in the illuminated region. There we of course expect the intensity to approach the intensity I_0 of the incident wave. In terms of the integral, we have

$$u_p(d = \infty) = \int_{-\infty}^{+\infty} d\eta e^{i\eta^2}$$

Right at the geometric boundary, we have

$$u_p(d = 0) = \int_{0}^{+\infty} d\eta e^{i\eta^2} = u_p(d = \infty)/2$$

From which we can deduce that

$$I(d = 0) = I_0/4$$

2. Intensity deep in the shadow

In this case we have to consider negative and large d, so that the integral becomes

$$u_p = \int_{|w|}^{+\infty} d\eta e^{i\eta^2} \qquad |w| = |d|\sqrt{\frac{k}{2D_p}}$$

Clearly, as $|d|$ becomes large this integral tends to zero and with it the intensity in the deep shadow.

How fast does $I(|d|)$ approach zero? To answer this question, we need to estimate the integral for large values of $|w|$. This can be done by the following per-parts integration:

$$\int_{|w|}^{+\infty} d\eta e^{i\eta^2} = \int_{|w|}^{+\infty} d\eta (e^{i\eta^2})' \frac{1}{2i\eta} = -e^{iw^2}\frac{1}{2i|w|} + \frac{1}{2i}\int_{|w|}^{+\infty} e^{i\eta^2}\frac{d\eta}{\eta^2}$$

For large $|w|$ the last integral can be neglected, and the first term is our amplitude estimate. From this, we conclude that the intensity is inversely proportional to the distance from the geometric light-shadow boundary:

$$I(|d|) \approx \frac{1}{d^2}$$

3. Intensity in the illuminated region

This time we need to find

$$u_p = \int_{-w}^{+\infty} d\eta e^{i\eta^2} \qquad w = d\sqrt{\frac{k}{2D_p}}$$

which we can do for large d using a similar trick as in the previous case. We only state the result of the approximation

$$u_p \approx \sqrt{i\pi} + \frac{1}{2iw}e^{iw^2} + \dots$$

The first term obviously represents the constant intensity I_0 for from the edge. The second will cause oscillation of the intensity value. The factor $1/w$ will damp the oscillation as we go further and further from the boundary. Since we have w^2 in the exponential, we can deduce that the oscillations are faster and faster further from the boundary.

Calculating the modulus square of the above leads us to this formula for the intensity in the illuminated region (but sufficiently far from the geometric light-shadow boundary):

$$I(w) = I_0 \left[1 + \sqrt{\frac{1}{\pi}} \frac{\sin(w^2 - \pi/4)}{w} \right]$$