Complex representation works exactly the same way as in 1D:

$$\psi(\vec{r},t) = A\cos[\vec{k}.\vec{r} - \omega t + p]$$

is replaced in calculation with a complex-valued

$$\psi_c(\vec{r},t) = A\exp[i(\vec{k}.\vec{r} - \omega t + p)]$$

The general rule is

$$\psi = Re\{\psi_c\}$$

Note: We often use the same notation for ψ and ψ_c. Context should tell us which one we work with at any particular moment.

Utility of the complex representation in calculations:

$$\begin{aligned} \nabla\psi &= +i\vec{k}\psi \\ \nabla^2\psi &= -k^2\psi \\ \frac{\partial\psi}{\partial t} &= -i\omega\psi \\ \frac{\partial^2\psi}{\partial t^2} &= -\omega^2\psi \end{aligned} \quad \rightarrow \quad \begin{aligned} \nabla &= +i\vec{k} \\ \nabla^2 &= -k^2 \\ \frac{\partial}{\partial t} &= -i\omega \\ \frac{\partial^2}{\partial t^2} &= -\omega^2 \end{aligned}$$

Note: This can only be applied in the complex representation, it is of course **not true** that e.g

$$\nabla\cos[\vec{k}.\vec{r}-\omega t] = \vec{k}\cos[\vec{k}.\vec{r}-\omega t]$$

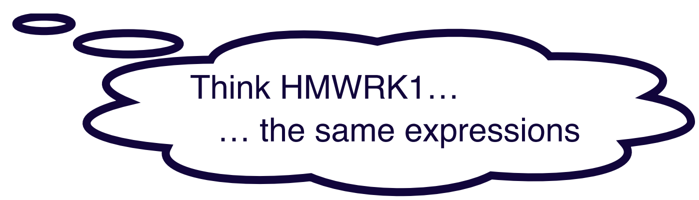
Note: Be careful when using the above operator "equivalencies." Check the material you work with for which concrete choice of plane-wave representation is used. Signs in these rules are not universal!

These rules can be expanded for **vector** plane waves. Consider

$$\vec{V}(\vec{r},t) = \vec{A}\exp[i(\vec{r}.\vec{k}-\omega t+p)] \equiv \hat{i}V_x + \hat{j}V_y + \hat{k}V_z$$

Check (by calculating component by component) that one can identify:

$$\begin{aligned} \nabla.\vec{V} &\equiv i\vec{k}.\vec{V} \\ \nabla\times\vec{V} &\equiv i\vec{k}\times\vec{V} \end{aligned}$$



Symmetry of a problem often suggests to work in **spherical coordinates** rather than in Cartesian system we have been using so far. Next we look at certain special solutions to 3D Wave Equation.

Restriction: Only consider radially symmetric functions/solutions

One such a solution is:

$$\psi(r, \theta, \phi, t) = \frac{A}{r} \cos[kr \pm \omega t + p]$$

where:

- there is no dependence of the RHS on either θ or ϕ
- as before, we will see that $k = \omega/c$
- the $\pm$ sign selects outward or inward propagating wave
- p is additional constant phase
- r is the distance from the origin, which is also the center of symmetry for this wave. Note that there is no dot product in the phase of cos!
- A measures the strength of the source, or intensity of the wave
- important difference from the plane waves is the $1/r$ prefactor — we shall see that it serves to conserve energy or radiated power

Spherical wave (now in complex representation) with a shifted center:

$$\psi(\vec{r},t) = \frac{A}{|\vec{r}-\vec{r}_0|}\exp[i(k|\vec{r}-\vec{r}_0| - \omega t)]$$

Q: Is this outgoing or inward propagating wave?

Exercise: Demonstrate that these spherical waves are indeed solutions to the wave equation.

- A) Do this in the Cartesian system of coordinates ...
- B) and show it using the wave equation specialized for radially symmetric functions (i.e. apply radially symmetric Laplacian)

Solution B:

The Laplacian operator in spherical coordinates reads:

$$\nabla^2 = \partial_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin\theta^2}\partial_{\phi\phi}\right]$$

but we only apply it to a function with no dependence on θ and ϕ, so the wave equation reads

$$\left(\partial_{rr} + \frac{2}{r}\partial_r - \frac{1}{v^2}\partial_{tt}\right)\frac{A}{r}\cos[kr \pm \omega t + p] = 0$$

The rest should be obvious: execute the derivatives, and collect terms. You will obtain a constrain for this being a solution — the dispersion relation $k^2 = \omega^2/v^2$.

$$\left(k^2 - \frac{\omega^2}{v^2}\right) \frac{A}{r} \cos[kr \pm \omega t + p] = 0$$

Note: Of course the result (that it *is* a solution) will be the same if one replaces cos for sin, or for $\exp[i(\ldots)]$.

Exercise: Obtain spherical waves by direct solution of the wave equation.

Answer: There if of course many different ways to do this, we will have a look at a method that transforms the equation to our known one-dimensional wave equation.

So our task is to solve

$$\left(\partial_{rr} + \frac{2}{r}\partial_r - \frac{1}{v^2}\partial_{tt}\right)\psi(r,t) = 0$$

Start with the radial Laplacian applied to ψ, and show that

$$(\partial_{rr} + \frac{2}{r}\partial_r)\psi = \psi_{rr} + 2/r\psi_r \equiv \frac{1}{r}\partial_{rr}(r\psi)$$

then you can write the WE as

$$\frac{1}{r}\partial_{rr}(r\psi) - \frac{1}{v^2}\partial_{tt}\psi$$

and multiply by r to get

$$\partial_{rr}(r\psi) - \frac{1}{v^2}\partial_{tt}(r\psi)$$

which is nothing but a one-dimensional wave equation for the function $(rpsi)$. So we can write e.g.

$$(r\psi) = A\cos[kr \pm \omega t + p]$$

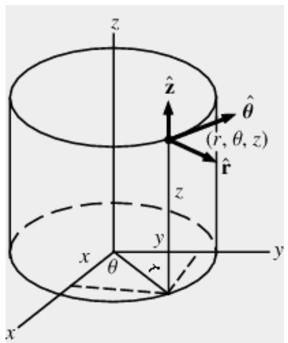
or

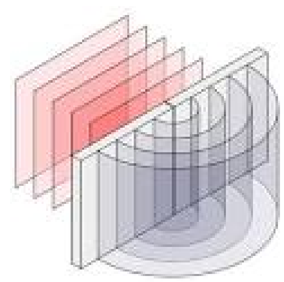
$$\psi = \frac{A}{r}\cos[kr \pm \omega t + p]\ .$$

The above solutions to WE are valid everywhere except the point at the origin, or center of the wave. This is its **source**.

Now we want to consider a situation when the source is a line (think of a narrow slit irradiated from one side with a plane wave). This is the case of **cylindrical waves**.

We will restrict ourselves to the axially symmetric case: All solutions will only depend on r, but not on z and ϕ in cylindrical coordinates. This means that the **axis z will be the source:**





Exercise:

Let the wave be given as

$$\psi(r,t) = \frac{A}{\sqrt{r}} \cos[kr \pm \omega t] \ .$$

where r is the radius in cylindrical coordinates (most of the time ρ is used instead).

Insert this into the wave equation in cylindrical coordinates and see if it can be a solution.

Answer: First we need Laplacian for cylindrical coordinates.It is

$$\nabla^2 = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\phi\phi} + \partial_{zz}$$

So we are testing if

$$\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{1}{v^2}\partial_{tt}\right)\psi(r,t)$$

equals zero.

Calculating what the radial part of the Laplacian gives, you should get

$$\left(\partial_{rr} + \frac{1}{r}\partial_r\right)\psi(r,t) = -k^2\psi + \frac{A\cos[kr \pm \omega t]}{r^{5/2}}$$

Inserting into wave equation we get

$$\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{1}{v^2}\partial_{tt}\right)\psi(r,t) = (\frac{\omega^2}{v^2} - k^2)\psi + \frac{A\cos[kr \pm \omega t]}{r^{5/2}}$$

... and we wanted zero on RHS. The first term is not offending, it is the usual solvability condition that fixes the dispersion relation between k, ω. The second term says that we do not have a solution. But this term decays quickly at a large distance r — this means that the function can approximate wave equation solution, as long as we do not use it around $r \approx 0$.

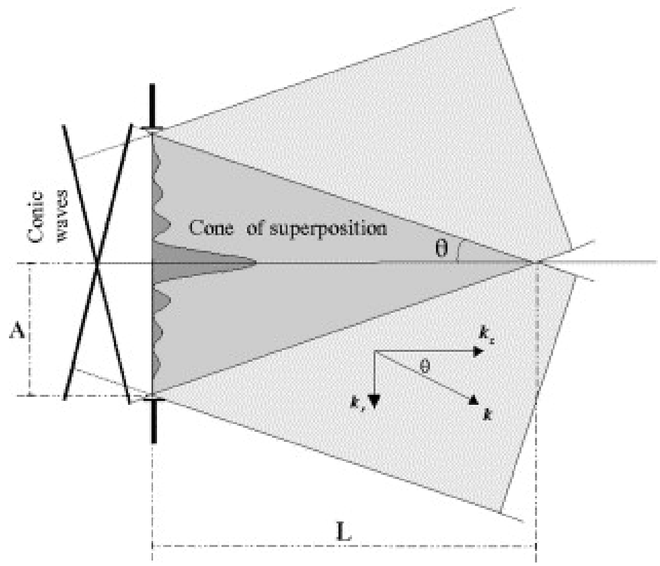
Note: In the previous example, there was no propagation in the z direction.

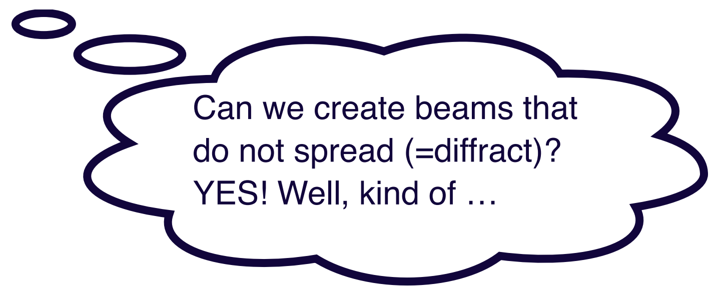
Q: Can you generalize the cylindrical wave such that it also propagates along the z-axis?

This is an **optional material**, beyond what is required for OPTI-310. It will shed more light on the cylindrical wave approximation and introduce the notion of a Bessel beam.

Consider a set of wave vector that together outline a cone:

$$\vec{k}(\phi) = \hat{k}k_z + \hat{j}k_\perp \sin\phi + \hat{i}k_\perp \cos\phi \quad \phi \in (0, 2\pi)$$





The angle of the cone is $\tan\theta = k_\perp / k_z$.

Next, assign a plane wave (of the same amplitude) to each of these wave vectors

$$\psi_\phi(\vec{r}) = \exp[i(\vec{k}(\phi).\vec{r} - \omega t)]$$

Now add all these plane wave into a superposition:

$$\psi = \int_0^{2\pi} d\phi \psi_\phi(\vec{r}) = \int_0^{2\pi} d\phi \exp[i(\vec{k}(\phi).\vec{r} - \omega t)]$$

$$\psi = \int_0^{2\pi} d\phi \exp[i(\vec{k}(\phi).\vec{r} - \omega t)] = \int_0^{2\pi} d\phi \exp[i(k_z z + k_\perp x \cos\phi + k_\perp y \sin\phi - \omega t)]$$

Symmetry of the problem suggests to use cylindrical coordinates for the result. So let us write $x = \rho\cos\alpha$ and $x = \rho\sin\alpha$, and pull out all "constants" outside of the integral:

$$\psi = \exp[i(k_z z - \omega t)] \int_0^{2\pi} d\phi \exp[ik_\perp \rho(\cos\alpha\cos\phi + \sin\alpha\sin\phi)]$$

$$\psi = \exp[i(k_z z - \omega t)] \int_0^{2\pi} d\phi \exp[ik_\perp \rho\cos(\phi - \alpha)]$$

This integral obviously does not depend on α so we can insert $\alpha = 0$ and get

$$\psi = \exp[i(k_z z - \omega t)] \int_0^{2\pi} d\phi \exp[ik_\perp \rho\cos\phi]$$

Apart from a multiplicative constant, this integral defines a well-know special function, namely Bessel J_0:

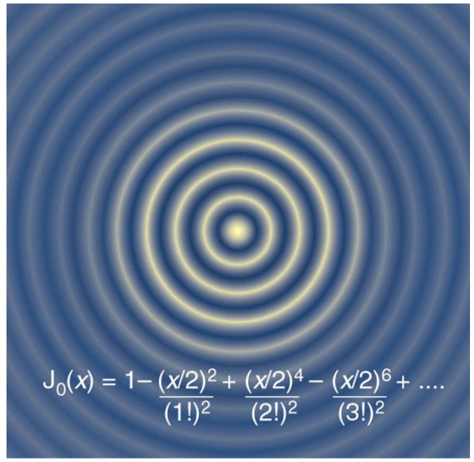
$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp[iz\cos\phi]$$

So our final result is

$$\psi(\rho, z, t) = \exp[i(k_z z - \omega t)] J_0(k_\perp \rho)$$

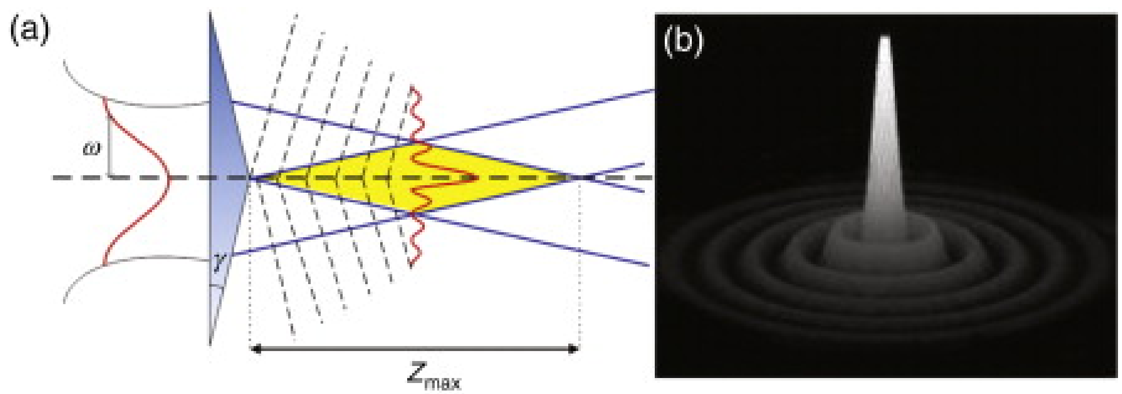
This is the scalar-wave version of the so-called Bessel beam. It propagates along z and exhibits infinitely many rings in its transverse cross-section.

Note: Bessel J_0 is the *envelope*, while the first exp term is the *carrier wave*.



This (and everything else)
is a superposition of PLANE WAVES

Transverse intensity cross-section of the Bessel beam. Each ring carries equal power.



This is how they are created in practice. Nobody has infinitely wide plane waves, of course. Here the conical superposition is approximated behind a conical lens, axicon, but only within certain zone close to optical axis.