OPTI-310. Review of useful formulas II.

Light-matter interactions

$$\vec{D} = \epsilon \vec{E} = \epsilon_0 \epsilon_r \vec{E} = \epsilon_0 n(\omega)^2 \vec{E} = \epsilon_0 \vec{E} + \vec{P}$$

Polarization (dipole moment density):

$$\vec{P} = N\vec{d} = N q_e \vec{x}(t)$$

Single oscillator model:

$$m_e \frac{d^2\vec{x}}{dt^2} = q_e \vec{E}(t) - m_e \omega_0^2 x(t)$$

$$n(\omega)^2 = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \frac{1}{\omega_0^2 - \omega^2}$$

relation to susceptibility

$$n(\omega)^2 = 1 + \chi(\omega)$$

Resonance frequency:

$$\hbar\omega_0 = E_e - E_g$$

Generalization 1: Mutiple oscillators, oscillator strengths

$$n(\omega)^2 = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2}$$

Generalization 2: Including damping

$$n(\omega)^2 = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 + i\omega\gamma_j}$$

Generalization 3: Condensed medium (oscillators experience each other's field):

$$\frac{n(\omega)^2 - 1}{n(\omega)^2 + 2} = \frac{N q_e^2}{3\epsilon_0 m_e} \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 + i\omega\gamma_j}$$

Generalization 4: Anisotropic oscillators for birefringent media

$$n_x(\omega)^2 = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \frac{1}{\omega_{0x}^2 - \omega^2}$$

Cubic, isotropic

$$n_x = n_y = n_x$$

Uniaxial

$$n_x = n_y \neq n_z$$

Biaxial

$$n_x \neq n_y \neq n_z \neq n_x$$

Cauchy formula:

$$n(\omega) = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + \dots$$

Sellmeier formula:

$$n^2(\omega) = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0j}^2}$$

Helmholtz equation

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}, \omega) e^{-i\omega t}$$

$$\left[\nabla^2 + k^2\right] \vec{E}(\vec{r}, \omega) = 0$$

Boundary conditions at material interfaces
Tangential componnets of intensities $\vec{E}$, $\vec{H}$ are continuous.
Normal componnets of inductions $\vec{D}$, $\vec{B}$ are continuous.

Reflection and refraction at a material interface

$$n_i \sin\Theta_i = n_t \sin\Theta_t \quad , \quad \Theta_i = \Theta_r$$

Fresnel formulae

(Note: vector orientation convention as in Hecht)
$\vec{E}$ normal to the plane of incidence, TE or s-polarization:

$$r_\perp = \frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} \quad , \quad t_\perp = \frac{2 n_i \cos\Theta_i}{n_i \cos\Theta_i + n_t \cos\Theta_t}$$

$\vec{E}$ in the plane of incidence, TM or p-polarization:

$$r_\parallel = \frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t} \quad , \quad t_\parallel = \frac{2 n_i \cos\Theta_i}{n_t \cos\Theta_i + n_i \cos\Theta_t}$$

Reflectance and transmittance, material and footprint factor

$$R = |r|^2 \quad , \quad T = \frac{n_t}{n_i} \frac{\cos\Theta_t}{\cos\Theta_i} |t|^2$$

Brewster angle:

$$\tan\Theta_{Brewster} = n \equiv n_t / n_i$$

Critical angle:

$$\sin\Theta_{Critical} = n$$

Other forms:

$$r_\perp = -\frac{\sin(\Theta_i - \Theta_t)}{\sin(\Theta_i + \Theta_t)} \quad , \quad t_\perp = +\frac{2\sin\Theta_t\cos\Theta_i}{\sin(\Theta_i + \Theta_t)}$$

$$r_\parallel = +\frac{\tan(\Theta_i - \Theta_t)}{\tan(\Theta_i + \Theta_t)} \quad , \quad t_\parallel = +\frac{2\sin\Theta_t\cos\Theta_i}{\sin(\Theta_i + \Theta_t)\cos(\Theta_i - \Theta_t)}$$

$$r_s = \frac{\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_s = \frac{2\cos\Theta_i}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

$$r_p = \frac{n^2\cos\Theta_i - \sqrt{n^2 - \sin^2\Theta_i}}{n^2\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}} \quad , \quad t_p = \frac{2n\cos\Theta_i}{\cos\Theta_i + \sqrt{n^2 - \sin^2\Theta_i}}$$

Normal incidence:

$$R = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2 \quad , \quad T = \frac{4n_t n_i}{(n_t + n_i)^2} \quad , \quad R + T = 1$$

Conversion: Hecht $\rightarrow$ Fowles

$$r_\perp \ , \ t_\perp \rightarrow r_s \ , \ t_s$$

$$r_\parallel \ , \ t_\parallel \rightarrow -r_p \ , \ t_p$$

Total internal reflection

$$r_s = \frac{\cos\Theta_i - i\sqrt{\sin^2\Theta_i - n^2}}{\cos\Theta_i + i\sqrt{\sin^2\Theta_i - n^2}} \quad , \quad \Theta_i > \Theta_{cr} \quad , \quad |r_s| = 1$$

Polarization

$$E_x(z,t) = A_x\cos[kz - \omega t + \phi_x] \quad , \quad E_y(z,t) = A_y\cos[kz - \omega t + \phi_y]$$

Implicit equation: Ellipse for the "tip" of the electric vector

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\left(\frac{E_x}{A_x}\right)\left(\frac{E_y}{A_y}\right)\cos\phi_{yx} = \sin^2\phi_{yx} \qquad \phi_{yx} = \phi_y - \phi_x$$

Ellipse angle

$$\tan 2\alpha = \frac{2A_X A_y\cos\phi_{yx}}{A_x^2 - A_y^2}$$

RHC,LHC

$$\vec{E} = A\hat{e}_\pm e^{ikz-\omega t} + c.c.$$

$$\hat{e}_{\pm}\frac{1}{\sqrt{2}}(\hat{x} \mp i\hat{y})$$

Jones vectors
general:

$$\mathcal{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

linear - x

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

linear - y

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

RHC

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

LHC

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$$

Jones matrices
Sequence of optical elements:

$$M = M_n \dots M_3 M_2 M_1$$

$$\mathcal{E}_{out} = M\mathcal{E}_{in}$$

Polarizer - x

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Polarizer - y

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Polarizer - ±45

$$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

Quarter waveplate, fast axis vertical, horizontal, ±45

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & +i \end{pmatrix} \quad , \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$$

Polarizer circular right, left

$$\frac{1}{2}\begin{pmatrix} 1 & +i \\ -i & 1 \end{pmatrix} \quad , \quad \frac{1}{2}\begin{pmatrix} 1 & -i \\ +i & 1 \end{pmatrix}$$

Rotator, rotating any (polarization-state) vector in positive, anti-clock-wise sense

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ +\sin\alpha & \cos\alpha \end{pmatrix}$$

Rotated elements

$$R(\alpha) = \begin{pmatrix} \cos\alpha & +\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$M = R^{-1}(\alpha)M'R(\alpha) = R(-\alpha)M'R(\alpha)$$

Rotated x-polarizer

$$M' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad M = \begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix}$$

Energy conservation and determinants, polarizers vs plates/retarders:

$$\det(M) = 0 \qquad \text{vs} \qquad |\det(M)| = 1$$