Name:	
Score:	/out of 50 possible points

OPTI 210, **<u>PRACTICE</u> Mid-Term Exam 2** Prof. M. Kolesik

Notes for the exam:

1. **This is a closed-book, closed-notes exam.** Calculators (with no text stored) may be used during the tests and final exam. No other form of electronic device may be used (no computers, laptops, PDA's, etc). Cell phones are absolutely prohibited during tests and the final exam. Food and drink are prohibited in the exams.

2. Answer all questions.

3. **Show your work and answers on the exam paper in the space following each question. Take the space available as a hint on how much you should be writing if you approach the problem correctly.** You may use additional paper if you find it necessary: this will be provided so do not bring your own paper into the exam. If you do use extra pages, staple the extra pages to the back of your exam. Make sure your final answers are clearly indicated.

4. On any sketches, make sure that axes are labeled and that important graphical trends are clear (such as amplitude, sign, or spatial considerations, etc.). If they are not clear enough, you may add a few words explaining what trends should be visible in the sketch.

5. Vector quantities should be distinguished by an overarrow such as $\vec{A}$.

CONSTANTS and FORMULAE of potential use in this exam:

$$r_\perp = \frac{n_i \cos\Theta_i - n_t \cos\Theta_t}{n_i \cos\Theta_i + n_t \cos\Theta_t} \quad , \quad t_\perp = \frac{2n_i \cos\Theta_i}{n_i \cos\Theta_i + n_t \cos\Theta_t}$$

$$r_\parallel = -\frac{n_t \cos\Theta_i - n_i \cos\Theta_t}{n_t \cos\Theta_i + n_i \cos\Theta_t} \quad , \quad t_\parallel = \frac{2n_i \cos\Theta_i}{n_t \cos\Theta_i + n_i \cos\Theta_t}$$

$$\sqrt{1+a} \approx 1 + \frac{1}{2}a + \dots$$

$$\frac{1}{v_?} = \frac{k(\omega)}{\omega} \qquad \frac{1}{v_?} = \frac{\partial k(\omega)}{\partial \omega}$$

P1: The Lorentz electron oscillator model for light-matter interactions leads to the following expression for the relative permittivity $\epsilon(\omega)$ experienced by a monochromatic field of angular temporal frequency ω propagating in a medium:

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{[(\omega_0^2 - \omega^2) - i\gamma\omega]} \qquad \text{where} \qquad \omega_p^2 = \frac{Nq^2}{\epsilon_0 m_e}$$

A model of for metals can be obtained from the above by considering a limit in which the resonance frequency vanishes:

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{[\omega^2 + i\gamma\omega]} \qquad \text{where} \qquad \omega_p^2 = \frac{Nq^2}{\epsilon_0 m_e}$$

The quantity ω_p is often called plasma frequency. The damping parameter γ can be interpreted as frequency of electron collisions. Both quantites have dimension of frequency. Typical values for metals are PHz and THz for plasma and collision frequency, respectively.

A) Consider two regimes: low and high frequency (in comparison with the plasma frequency) for a material in which collisions can be neglected. By extracting the refractive index from permittivity, give an argument that metals are transparent at high frequencies.

B) Now consider the case of frequency ω much lower than plasma. Using the refractive index in the Fresnel formulas, show that the metal becomes highly reflective.

C) For silver, the plasma frequency is about $\omega_p \approx 1 \times 10^{16}\mathrm{s}^{-1}$. Estimate the density of free electrons.

D) For the collision frequency in silver given as $\gamma = 1 \times 10^{14}$ Hz, estimate the reflection coefficient at normal incidence for the wavelength of $\lambda = 800\mathrm{nm}$.

Note: The model of silver as given here is oversimplified, and not very realistic. Yet, it captures the basic trends in the optical properties.

Answers:

A)

$$n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \sim 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$$

because the second term is much smaller than unity, the square root is real-valued, refractive index has zero imaginary part, and therefore there is no optical losses, which in turn means that the medium appears transparent.

B)

$$n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \sim i\frac{\omega_p}{\omega}$$

because the fraction under $\sqrt{}$ dominates 1.

Now calculate reflection coeff (use Fresnel for normal incidence):

$$r = \frac{1 - i\frac{\omega_p}{\omega}}{1 + i\frac{\omega_p}{\omega}} \qquad |r| = 1$$

because r is ratio of two complex numbers that are conjugates of each other, it is a pure phase (something like $\exp[i\phi]$) and its absolute value is one. This means that all light is reflected.

C) Insert constants into:

$$N = \frac{\omega_p^2 \epsilon_0 m_e}{q^2} \sim 10^{29}$$

where rather rough estimate is all that we need...

D) First figure out ω for given λ: it is about 2.3×10^{15}/s. Insert, together with given γ into $\epsilon(\omega)$. Evaluate. You should obtain a value in which the real part dominates and is negative (about -25). The corresponding refractive index is then imaginary, so the reflection coeff has absolute value close to unity. This is the same regime as in B).

P2:
This problem concernes properties of Brewster windows. Let the material refractive index be $\sqrt{2}$.

A) Given Fresnel formulas, derive the condition the Brewster angle of incidence must satisfy.

B) In you wish to place a Brewster plate in a laser cavity to promote certain linear polarization, how should the plate be oriented?

C) Evaluate the amplitude *reflection* coefficients for both polarizations at the two interfaces the beam passes through. Express the result solely in terms of n (i.e. your result should not contain any angles, and should not be evaluated for the given value of n).

D) If the incident light is unpolarized (natural), what fraction of power (or energy) is reflected from the first surface of the Brewster window?

E) Every partially polarized wave can be written as a superposition of an unpolarized wave plus a fully polarized wave. Degree of polarization (DOP) is calculated as the fraction of the total power that is carried by the polarized component of the wave. Calculate DOP behind the (single) interface for unpolarized light at Brewster incidence. Express the result as a function of the refractive index n.

Answers:

A) Derivation alternative to that in notes (using alternative form of Fresnel — see compendium):

$$0 = r_p = -\frac{\tan(\Theta_i - \Theta_p)}{\tan(\Theta_i + \Theta_p)} \rightarrow \Theta_i + \Theta_p = \pi/2$$

in other words, sum $\Theta_i + \Theta_p$ must be $\pi/2$ in order for tangent to diverge and make r_p to vanish. The fact that the two angles are complementary means:

$$\sin\Theta_t = \cos\Theta_i$$

Use this in in the Snell1' law:

$$n_i \sin\Theta_i = n_t \sin\Theta_t = n_t \cos\Theta_t \rightarrow \tan\Theta_i = n_t/n_i$$

B) The lasing polarization must correspond to p-polarization, because that is one that goes through the plate without loss. That means that the propagation direction (=cavity axis, $\hat{z}$) and polarization direction ($\hat{x}$), must constitute the plane of incidence. Normal $\hat{n}$ of the window must therefore lie in $x - z$ plane. To ensure that angle of incidence is Θ_B, choose:

$$\hat{n} = (\sin\Theta_B, 0, \cos\Theta_B)$$

and express in term of refractive index, by utilizing the formulae:

$$\sin\Theta_B = \frac{\tan\Theta_B}{\sqrt{1 + (\tan\Theta_B)^2}} = \frac{n}{\sqrt{1+n^2}}$$

and

$$\cos\Theta_B = \frac{1}{\sqrt{1 + (\tan\Theta_B)^2}} = \frac{1}{\sqrt{1+n^2}}$$

... to get

$$\hat{n} = (\frac{n}{\sqrt{1+n^2}}, 0, \frac{1}{\sqrt{1+n^2}})$$

C) Using the above formulas, find

$$\cos\Theta_t = \frac{n}{\sqrt{1+n^2}}$$

and use in Fresnel. As a sanity check, r_p must vanish, and

$$r_s = \frac{1-n^2}{1+n^2} = -1/3$$

D)

$$P/P_0 = |r_s|^2 = 1/9$$

D) Split the light into s and p, calculate fraction of power in each to get through:

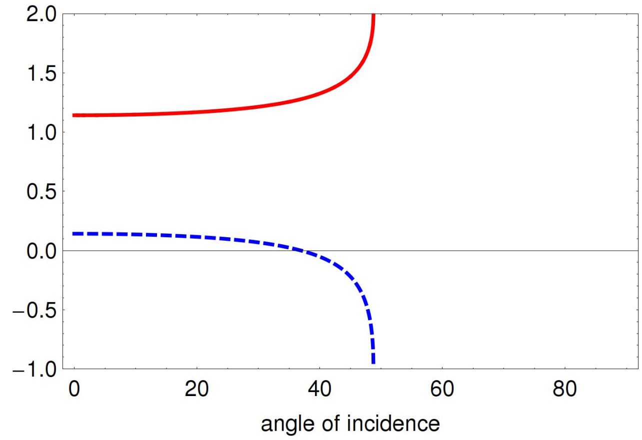
$$f_s = 1/2(1 - r_s^2) \qquad f_p = 1/2 = 1/2(1 - r_s^2) + 1/2r_s^2$$

Identify the "common" part in both - that is the unpolarized component. The rest is polarized. Construct the DOP:

$$DOP = \frac{1/2r_s^2}{1/2(1-r_s^2) + 1/2(1-r_s^2) + 1/2r_s^2} = \frac{1/2r_s^2}{1 - 1/2r_s^2} = \frac{r_s^2}{2 - r_s^2} = \frac{1/9}{2 - 1/9} = \frac{1/9}{17/9} = 1/17$$

Sanity check: DOP must be small, because only a small fraction of light is removed by reflection from what was initially unpolarized light.

P4: This problem deals with Fresnel formulas.



A: Does this picture show internal or external incidence? Why?

B: Which of the quantities r_s, r_p, t_s, t_p quantities are shown? Specify and justify your answer for each curve.

C: Show the critical angle in the picture, and estimate its value. Do the same for the Brewster angle.

D: Taking your estimate of the Brewster angle, give an estimate for the refractive index of the medium.

E: If you know that the red curve starts at normal incidence from the value of 1.2, how can you calculate the refractive index?

Answers:

A) Internal, because vertical asymptote implies critical angle.

B)

$$t_s \quad r_p$$

for red and blue, respectively.

C) Brewster is where r_p crosses zero. Critical is where the verical asymptote is located.

D) use $\tan \Theta_B = n$, concrete value not too important as long as the right formula is used.

E) Use Fresnel value at normal incidence

$$t_s(\Theta = 0) = 1.2 = \frac{6}{5} = \frac{2n}{1+n} \rightarrow n = 1.5$$