## Cross products and different ways to calculate them

In OPTI-210, cross-products appear in many contexts, and it is therefore good to practice different ways to evaluate them. Several methods are illustrated in what follows, each best suitable for certain given inputs.

Calculate $\vec{c} = \vec{a} \times \vec{b}$ for given $\vec{a} = (1, 2, 2)$ and $\vec{b} = (4, 3, 2)$

### 1. Method: Per definition

Inserting given inputs into the the "definition" formula(s):

$$\begin{aligned} c_x &= a_y * b_z - a_z * b_y = 2 * 2 - 2 * 3 = -2 \\ c_y &= a_z * b_x - a_x * b_z = 2 * 4 - 1 * 2 = +6 \\ c_z &= a_x * b_y - a_y * b_x = 1 * 3 - 2 * 4 = -5 \end{aligned}$$

$$\vec{c} = (-2, 6, -5)$$

This method is straightforward, but error-prone with numbers. It is better suited for derivations and for work with symbolic expressions. We shall utilize this extensively with Maxwell equations...

### 2. Determinant method,

is best suitable when the given vectors are "full" and numerical. It makes it easier to keep track of "what is multiplying what."

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 4 & 3 & 2 \end{vmatrix} = \hat{i} * 2 * 2 + \hat{j} * 2 * 4 + \hat{k} * 1 * 3 - \hat{i} * 2 * 3 - \hat{j} * 1 * 2 - \hat{k} * 2 * 4$$

$$\vec{c} = -2\hat{i} + 6\hat{j} - 5\hat{k}$$

### 3. Simplify first method

Recall that the portion of $\vec{b}$ that is colinear with $\vec{a}$ does not contribute to their cross-product. You can therefore subtract from each vector something proportional to the other vector in order to reduce number of non-zero components:

$$\vec{c} = \vec{a} \times \vec{b} = (1, 2, 2) \times (4, 3, 2) = (1, 2, 2) \times ((4, 3, 2) - (1, 2, 2)) = (1, 2, 2) \times (3, 1, 0)$$

You can go further and eliminate one element in $\vec{a}$ in a same way, by subtracting $\vec{a} \to \vec{a} - 2(3, 1, 0)$:

$$\vec{c} = (1, 2, 2) \times (3, 1, 0) = (-5, 0, 2) \times (3, 1, 0)$$

Now that there are two zeros, it is easier to read-off the result because there is only a single term contributing to each component:

$$c_x = -1 * 2 \quad c_y = 2 * 3 \quad c_z = -5 * 1 \; .$$

This method is sometimes efficient in symbolic calculations, especially if the geometry is simple, e.g. if $\vec{a} = (\cos\theta, \sin\theta, 0)$, $\vec{b} = (\cos\theta, \sin\theta, 1)$.

**4. Algebraic method**
is suitable if the given vectors are "sparse." Say we have already simplified first, and now have

$$\vec{c} = (-5, 0, 2) \times (3, 1, 0) \; .$$

Write both vectors in terms of $\hat{\imath}, \hat{\jmath}, \hat{k}$ and expand:

$$\vec{c} = (-5\hat{\imath} + 2\hat{k}) \times (3\hat{\imath} + \hat{\jmath}) = 2 * 3 * \hat{k} \times \hat{\imath} + 2 * 1 * \hat{k} \times \hat{\jmath} - 5 * 1 * \hat{\imath} \times \hat{\jmath} = 6\hat{\jmath} - 2\hat{\imath} - 5\hat{k} \; ,$$

where we used the algeraic rules for vector products between $\hat{\imath}, \hat{\jmath}, \hat{k}$.

**5. Geometry method**
In some cases the geometry of the problem is so simple that it is easy to guess what the cross product gives. For example, let us take $\vec{c} = (0, 0, 2) \times (3, 1, 0)$.

a) Figure out direction
b) Figure out magnitude
c) Check the Right-Hand-System orientation, and flip sign if not RHS

a)
Form the first vector we know that $\vec{c}$ does not have any $z$-component. It must lie in plane $(x, y)$. At the same time, it must be perpendicular to $(3, 1, 0)$. One particular vector with that property is $(-1, 3, 0)$. The answer must be proportional to this.

b)
We can see that the two vectors are perpendicular to each other ($\sin\theta = 1$). That means that the magnitude $|\vec{c}|$ must be $2|(3, 1, 0)| = 2|(-1, 3, 0)|$. That is how we know that $\vec{c} = \pm 2(-1, 3, 0)$.

c)
To fix the sign, make sure that the three vectors constitute a right-hand-oriented system. This is probably easiest with a sketch. A drawing will readily tell you that the result is

$$\vec{c} = +2(-1, 3, 0)$$

**Recommendation**

Use methods 2 or 4 for calculation, and use method 5 (if possible) to sanity-check the result.