Beyond OPTI-210: Vector algebra in component from

These notes illustrate yet another (extremely useful) way to manipulate vector expressions. The method works with components and that is why it is often called 'component method.' It has the distinct advantage that one does not need to remember any vector identities except one formula for a product consisting of two "matrices" representing the so-called Levi-Civita symbol. With this, the calculations are straightforward, and require little thinking beyond the setup of the initial expression. The method is also easy to program/automate...

There are a couple of objects and rules to manipulate them:

1 **Kronecker delta** tensor, is essentially an identity matrix. It has two indices and values zero or unity, as follows:

$$\delta_{ij} = 1 \quad \text{for} \quad i = j \quad \text{and} \quad \delta_{ij} = 0 \quad \text{for} \quad i \neq j$$

where each index can attain values from 1 up to the dimension of space we work in (i.e. three in our case).

2 **Levi-Civita symbol.** It is an object with three indices that can be either zero or plus or minus one:

$$\epsilon_{ijk} = 0 \quad \text{if any two indices are equal, for example} \quad \epsilon_{223} = 0$$

$$\epsilon_{ijk} = +1 \quad \text{if ijk is even permutation of 123}$$

$$\epsilon_{ijk} = -1 \quad \text{if ijk is odd permutation of 123}$$

3 **Einstein's summation rule:** We leave out summation signs. One can easily recognize what is summed up in the expressions because such sums always occur over pairs of repeated indices. For example a matrix product:

$$C_{jk} = \sum_{i=1,2,3} A_{ji} B_{ik} \quad \text{is written simply as} \quad C_{jk} = A_{ji} B_{ik}$$

with the summation over i implied. With this rule, the action of Kronecker on an arbitrary vector is like this

$$\delta_{ij} a_j = a_i$$

4 Vector (cross) product and scalar product:

$$\vec{a}.\vec{b} \qquad \text{can be written as (the name of index used is arbitrary)} \qquad a_i b_i$$

$$\vec{c} = \vec{a} \times \vec{b} \qquad \text{can be written with Levi-Civita as} \qquad c_i = \epsilon_{ijk} a_j b_k$$

Note: convince yourself that it is actually true!

5 One last thing we use (the only one we need to remember) is the contracted epsilon identity:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Now we are ready for calculations. As an example, prove the identity we discussed in the class:

$$w = a \times (b \times c) = b(a.c) - c(a.b)$$

Write the symbol for left-hand side:

$$w_i = \epsilon_{ijk} a_j (b \times c)_k = \epsilon_{ijk} a_j \epsilon_{kmn} b_m c_n = \epsilon_{kij} \epsilon_{kmn} a_j b_m c_n$$

where we rotated indices in the first epsilon, and now use the contracted epsilon identity to get

$$w_i = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_j b_m c_n$$

where you apply Kronecker to the vectors with the same index:

$$w_i = b_i a_n c_n - c_i a_m b_m$$

In this expression, the repeated indices result in scalar products, and the free index i is the vector component of the result. So we can rewrite this back to vector symbols as

$$w = b(a.c) - c(a.b)$$

which is what we have on the other side of the identity we just proved.

Your turn:

• Prove that

$$(a \times b).(c \times d) = \begin{vmatrix} a.c & b.c \\ a.d & b.d \end{vmatrix}$$

• The technique is especially useful when some of the vectors are differential operators. Use it to prove the identity we shall need to obtain wave equations from Maxwell equations:

$$\nabla \times \nabla \times \vec{A} = \nabla \nabla . \vec{A} - \Delta \vec{A}$$